1. $\mathbf{r}(t)$ is the position of a particle in space at time t. Find the particle's velocity and acceleration vectors. Then find the particle's speed and direction of motion. Show your work. (8 points)

$$\mathbf{r}(t) = (e^{-t})\mathbf{i} + (2\cos 3t)\mathbf{j} + (2\sin 3t)\mathbf{k}, \qquad t = 0$$

$$v(t) = r'(t) = (-e^{-t})i - (6\sin 3t)j + (6\cos 3t)k$$

$$\boxed{v(0) = r'(0) = \langle -1, 0, 6 \rangle}$$

$$a(t) = v'(t) = (e^{t})i - (18\cos 3t)j - (18\sin 3t)k$$

$$\boxed{a(0) = v'(0) = \langle 1, -18, 0 \rangle}$$

$$\text{Speed at } t=0 = |v(0)| = \sqrt{(-1)^2+0^2+6^2} = \boxed{\sqrt{37}}$$

$$\text{Direction of motion} = \frac{v(0)}{|v(0)|} = \boxed{\left\langle -\frac{1}{\sqrt{37}}, 0, \frac{6}{\sqrt{37}} \right\rangle}$$

2. Solve the initial value problem for $\mathbf{r}$ as a vector of t. Show your work. (6 points)

$$\frac{d\mathbf{r}}{dt} = (t^3 + 4t)\mathbf{i} + t\mathbf{j} + 2t^2\mathbf{k}, \qquad \mathbf{r}(0) = \mathbf{i} + \mathbf{j}$$

$$dr = \left[(t^3+4t)i + tj + 2t^2k\right]dt$$

$$\int dr = \int \left[(t^3+4t)i + tj + 2t^2k\right]dt$$

$$r(t) = \left(\frac{t^4}{4} + 2t^2\right)i + \left(\frac{t^2}{2}\right)j + \frac{2t^3}{3}k + C$$

$$r(0) = C = i + j$$

$$r(t) = \left(\frac{t^4}{4} + 2t^2\right)i + \left(\frac{t^2}{2}\right)j + \frac{2t^3}{3}k + i + j$$

$$\boxed{r(t) = \left(\frac{t^4}{4} + 2t^2 + 1\right)i + \left(\frac{t^2}{2} + 1\right)j + \frac{2t^3}{3}k}$$

3. Find the muzzle speed of a gun whose maximum range is 28,000 m. Show your work. (Hint: For what angle is the range maximum?) (6 points)

Range $= 28{,}000 = \dfrac{v_0^2 \sin 2\alpha}{g}$ is maximum when $\sin 2\alpha = 1$

$\alpha = 45^\circ$

$\therefore\ 28{,}000 = \dfrac{v_0^2}{g}$

$v_0^2 = 28{,}000 \times 9{\cdot}8 = 274{,}400$

$v_0 = \sqrt{274{,}400}$ m/sec $= \boxed{523{\cdot}83 \text{ m/sec}}$

4. Find the length of the indicated portion of the curve. Show your work. (8 points)

$$\mathbf{r}(t) = (\cos^3 t)\mathbf{j} + (\sin^3 t)\mathbf{k}, \quad 0 \leq t \leq \pi/2$$

$r'(t) = \left(-3\cos^2 t \sin t\right)j + \left(3\sin^2 t \cos t\right)k$

$|r'(t)| = \sqrt{9\cos^4 t \sin^2 t + 9\sin^4 t \cos^2 t + 0}$

$= \sqrt{9\cos^2 t \sin^2 t(\cos^2 t + \sin^2 t)} = \sqrt{9\cos^2 t \sin^2 t}$

$= 3\cos t \sin t$

$L = \displaystyle\int_0^{\pi/2} |r'(t)|\, dt = \int_0^{\pi/2} 3\cos t \sin t\, dt$

$= \dfrac{3}{2}\displaystyle\int_0^{\pi/2} 2\cos t \sin t\, dt$

$= \dfrac{3}{2}\displaystyle\int_0^{\pi/2} \sin 2t\, dt = -\dfrac{3}{4}\cos 2t\Big|_0^{\pi/2} = \boxed{\dfrac{3}{2}}$

5. Find the limit. Show your work. (6 points)

$$\lim_{\substack{(x,y)\to(1,1)\\ x\neq 1}} \frac{xy - y - 3x + 3}{x - 1}$$

$$= \lim_{(x,y)\to(1,1)} \frac{y(x-1) - 3(x-1)}{(x-1)}$$

$$= \lim_{(x,y)\to(1,1)} \frac{(y-3)\cancel{(x-1)}}{\cancel{(x-1)}}$$

$$= \lim_{(x,y)\to(1,1)} y - 3 = 1 - 3 = \boxed{-2}$$

6. By considering different paths of approach, show that the function has no limit as $(x, y) \to (0, 0)$. Show your work. (6 points)

$$f(x, y) = \frac{x^4 - y^2}{x^4 + y^2}$$

Take $y = mx^2$

$$f(x, mx^2) = \frac{x^4 - m^2x^4}{x^4 + m^2x^4} = \frac{x^4(1-m^2)}{x^4(1+m^2)}$$

$$\lim_{(x,y)\to 0} \frac{x^4 - y^2}{x^4 + y^2} = \lim_{x\to 0} \frac{\cancel{x^4}(1-m^2)}{\cancel{x^4}(1+m^2)} = \frac{1-m^2}{1+m^2}$$

which has different values depending on m.

Hence $f(x,y)$ has no limit.

7. Find $\partial f/\partial x$ and $\partial f/\partial y$. Show your work. (8 points)

$$f(x,y) = e^{xy}\ln x$$

$$\boxed{\frac{\partial f}{\partial x} = e^{xy}\cdot\frac{1}{x} + \ln x\cdot e^{xy}\cdot y}$$

$$\boxed{\frac{\partial f}{\partial y} = \ln x\, e^{xy}\cdot x}$$

8. Verify that $w_{xy} = w_{yx}$. Show your work. (6 points)

$$w = e^x + x\ln y + y\ln x$$

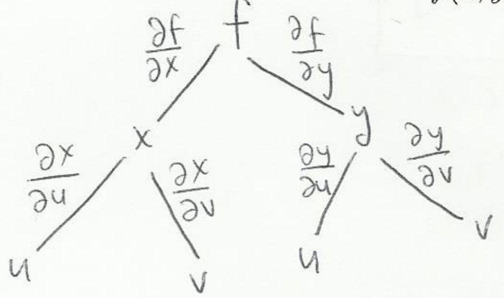
$w_x = e^x + \ln y + \frac{y}{x}$, $w_y = \frac{x}{y} + \ln x$

$w_{xy} = \frac{1}{y} + \frac{1}{x}$, $w_{yx} = \frac{1}{y} + \frac{1}{x}$

$\therefore \boxed{w_{xy} = w_{yx}}$

9. Draw a branch diagram and write a Chain Rule formula for $\partial z/\partial u$ and $\partial z/\partial v$. Show your work. (6 points)

$$z = f(x, y), \quad x = g(u, v), \quad y = h(u, v).$$



$$\boxed{\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u}}$$

$$\boxed{\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v}}$$

10. Use the above result to find $\partial z/\partial u$ when $u = 0, v = 1$ if $z = \sin xy + x \sin y$, $x = u^2 + v^2, y = uv$. Show your work. (8 points)

$$\frac{\partial z}{\partial x} = y.\cos xy + \sin y \quad , \quad \frac{\partial z}{\partial y} = x \cos(xy) + x \cos y$$

$$\frac{\partial x}{\partial u} = 2u \quad , \quad \frac{\partial y}{\partial u} = v$$

$$\left.\frac{\partial z}{\partial u}\right|_{\substack{u=0\\v=1}} = \left.(y \cos xy + \sin y)\right|_{\substack{u=0\\v=1}} . 2.0 + \left.(x \cos(xy) + x \cos y)\right|_{\substack{u=0\\v=1}} . 1$$

When $u=0, v=1$

$$x = 0^2 + 1^2 = 1$$

$$y = 0 \cdot 1 = 0$$

$$\therefore \left.\frac{\partial z}{\partial y}\right|_{\substack{u=0\\v=1}} = 1 . \cos 0 + 1 . \cos 0 = 2$$

$$\therefore \left.\frac{\partial z}{\partial u}\right|_{\substack{u=0\\v=1}} = 0 + 2 \cdot 1 = \boxed{2}$$

11. Find ∇f. Then find the derivative of the function f at P_0 in the direction of $\mathbf{v}$. Show your work. (8 points)

$$f(x,y,z) = xy + yz + zx, \quad P_0(1,-1,2), \quad \mathbf{v} = 3\mathbf{i} + 6\mathbf{j} - 2\mathbf{k}$$

$\nabla f = \langle y+z,\ x+z,\ x+y \rangle$

$\nabla f(1,-1,2) = \langle -1+2,\ 1+2,\ 1-1 \rangle = \langle 1,3,0 \rangle$

$u = \dfrac{v}{|v|} = \dfrac{\langle 3,6,-2 \rangle}{\sqrt{9+36+4}} = \langle \frac{3}{7}, \frac{6}{7}, -\frac{2}{7} \rangle$

$\therefore\ D_u f\big|_{P_0} = \nabla f(1,-1,2) \cdot u = \langle 1,3,0 \rangle \cdot \langle \frac{3}{7}, \frac{6}{7}, -\frac{2}{7} \rangle$

$= \frac{3}{7} + \frac{18}{7} - 0$

$= \frac{21}{7} = \boxed{3}$

12. Find the equations of the (a) tangent plane and (b) normal line at the point P_0 on the given surface. Show your work. (8 points)

$$x^2 + 2xy - y^2 + z^2 = 7, \quad P_0(1,-1,3)$$

(a) $f_x = 2x + 2y$, $f_y = 2x - 2y$, $f_z = 2z$

$f_x(1,-1,3) = 2 - 2 = 0$, $f_y(1,-1,3) = 2 +$ [redacted] $= 4$, $f_z(1,-1,3) = 6$

Tangent plane

$0(x-1) + 4(y+1) + 6(z-3) = 0$

$4y + 4 + 6z - 18 = 0$

$\boxed{4y + 6z = 14}$ or $\boxed{2y + 3z = 7}$

(b) $x = 1 + 0 \cdot t =$ [redacted] 1

$y = -1 + 4 \cdot t = -1 + 4t$

$z = 3 + 6 \cdot t = 3 + 6t$

13. Find the local maxima, local minima, and saddle points of the function. Show your work. (8 points)

$$f(x, y) = 6x^2 - 2x^3 + 3y^2 + 6xy$$

$f_x = 12x - 6x^2 + 6y = 0$ $\qquad f_{xx} = 12 - 12x$

$f_y = 6y + 6x = 0$ $\qquad f_{yy} = 6$

$\Rightarrow y = -x$ $\qquad f_{xy} = 6$

$\therefore 12x - 6x^2 - 6x = 0$

$6x - 6x^2 = 0$

$x(1-x) = 0$

$x = 0, 1$

$\Rightarrow y = 0, -1$

Critical points are $(0,0)$ and $(1,-1)$

$\underline{(0,0)}$

$D = f_{xx}(0,0) f_{yy}(0,0) - f_{xy}(0,0)^2$

$= 12 \cdot 6 - 6^2$

$= 72 - 36 = 36 > 0$

and $f_{xx}(0,0) = 12 > 0$

$\therefore$ $(0,0)$ is a point of local minima.

$f(0,0) = 0$ is the local minima.

$\underline{(1,-1)}$

$D = f_{xx}(1,-1) \cdot f_{yy}(1,-1) - f_{xy}(1,-1)^2$

$= 0 \cdot 6 - 6^2 = -36 < 0$

$\therefore$ $(1,-1)$ is a saddle point.

14. Using the method of Lagrange multipliers, find the points on the sphere $x^2+y^2+z^2 = 25$ where $f(x, y, z) = x+2y+3z$ has its maximum and minimum values. Show your work. (8 points)

$$g(x,y,z) = x^2+y^2+z^2-25 = 0$$
$$f(x,y,z) = x+2y+3z$$
$$\nabla f = \langle 1,2,3 \rangle \quad , \quad \nabla g = \langle 2x, 2y, 2z \rangle$$
$$\nabla f = \lambda \nabla g = \langle 1,2,3 \rangle = \langle 2x\lambda, 2y\lambda, 2z\lambda \rangle$$
$$\Rightarrow \quad 2x\lambda = 1 \,, \quad 2y\lambda = 2, \quad 2z\lambda = 3$$
$$\Rightarrow \quad x = \frac{1}{2\lambda}, \quad y = \frac{2}{2\lambda}, \quad z = \frac{3}{2\lambda}$$

$$x^2+y^2+z^2 = 25$$
$$\left(\frac{1}{2\lambda}\right)^2 + \left(\frac{2}{2\lambda}\right)^2 + \left(\frac{3}{2\lambda}\right)^2 = 25$$
$$\frac{14}{4\lambda^2} = 25 \Rightarrow \lambda^2 = \frac{14}{100}$$
$$\lambda = \pm\frac{\sqrt{14}}{10}$$

$\therefore$ $x = \frac{1}{2\lambda} = \pm\frac{5}{\sqrt{14}}$

$y = \frac{2}{2\lambda} = \pm\frac{10}{\sqrt{14}}$

$z = \frac{3}{2\lambda} = \pm\frac{15}{\sqrt{14}}$

$\therefore$ Maximum is attained at the point $\left(\frac{5}{\sqrt{14}}, \frac{10}{\sqrt{14}}, \frac{15}{\sqrt{14}}\right)$

Minimum is attained at the point $\left(-\frac{5}{\sqrt{14}}, -\frac{10}{\sqrt{14}}, -\frac{15}{\sqrt{14}}\right)$