1. Evaluate the double integral over the given region R. Show your work. (8 points)

$$\iint_R y\sin(x+y)\,dA, \quad R: \quad -\pi \le x \le 0, \ \ 0 \le y \le \pi$$

$$\int_{-\pi}^{0}\int_{0}^{\pi} y\sin(x+y)\,dy\,dx$$

$u = y \quad dv = \sin(x+y)dy$

$du = dy \quad v = -\cos(x+y)$

$$= \int_{-\pi}^{0}\left[-y\cos(x+y)\Big|_0^{\pi} + \int_0^{\pi}\cos(x+y)dy\right]dx$$

$$= \int_{-\pi}^{0}\left[-\pi\cos(x+\pi) + \sin(x+y)\Big|_0^{\pi}\right]dx$$

$$= \int_{-\pi}^{0}\left[\pi\cos x + \sin(x+\pi) - \sin x\right]dx$$

$$= \int_{-\pi}^{0}\left[\pi\cos x - \sin x - \sin x\right]dx$$

$$= \int_{-\pi}^{0}\pi\cos x\,dx - \int_{-\pi}^{0}2\sin x\,dx$$

$$= \pi\sin x\Big|_{-\pi}^{0} + 2\cos x\Big|_{-\pi}^{0}$$

$$= 2\cos 0 - 2\cos(-\pi)$$

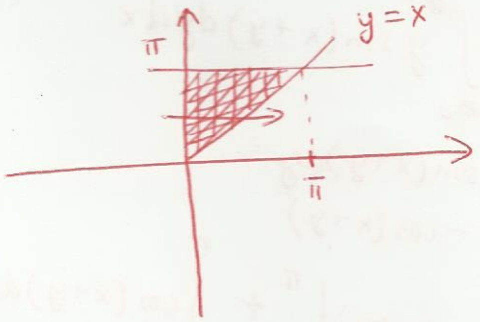
$$= 2 + 2 = \boxed{4}$$

2. Sketch the region of integration, reverse the order of integration, and evaluate the integral. Show your work. (8 points)

$$\int_0^{\pi}\int_x^{\pi} \frac{\sin y}{y}\, dy\, dx$$

$x \leq y \leq \pi$

$0 \leq x \leq \pi$



$0 \leq x \leq y$

$0 \leq y \leq \pi$

$$\int_0^{\pi}\int_0^{y} \frac{\sin y}{y}\, dx\, dy = \int_0^{\pi} \frac{x \sin y}{y}\Big|_0^y dy = \int_0^{\pi} \frac{\cancel{y}\sin y}{\cancel{y}}\, dy$$

$$= \int_0^{\pi} \sin y\, dy$$

$$= -\cos y \Big|_0^{\pi}$$

$$= -\cos\pi + \cos 0 = 1+1 = \boxed{2}$$

3. Change the Cartesian integral into an equivalent polar integral. Then evaluate the integral. (8 points)

$$\int_0^1 \int_x^{\sqrt{2-x^2}} (x+2y)\,dy\,dx$$

$x \le y \le \sqrt{2-x^2}$

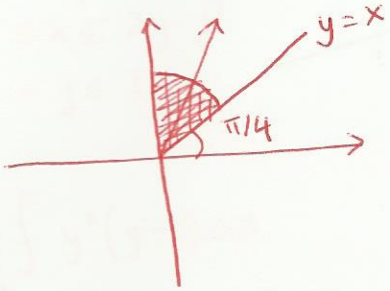
$0 \le x \le 1$

$y \le \sqrt{2-x^2}$

$y^2 \le 2-x^2$

$x^2+y^2 \le 2$

$r \le \sqrt{2}$



$0 \le r \le \sqrt{2}$ $\quad \frac{\pi}{4} \le \theta \le \frac{\pi}{2}$

$$\int_{\pi/4}^{\pi/2}\int_0^{\sqrt{2}} (r\cos\theta + 2r\sin\theta)\, r\,dr\,d\theta = \int_{\pi/4}^{\pi/2} \frac{r^3}{3}\cos\theta + \frac{2r^3}{3}\sin\theta \Big|_0^{\sqrt{2}} d\theta$$

$$= \int_{\pi/4}^{\pi/2} \left(\frac{2\sqrt{2}}{3}\cos\theta + \frac{4\sqrt{2}}{3}\sin\theta\right) d\theta$$

$$= \frac{2\sqrt{2}}{3}\sin\theta \Big|_{\pi/4}^{\pi/2} - \frac{4\sqrt{2}}{3}\cos\theta \Big|_{\pi/4}^{\pi/2}$$

$$= \frac{2\sqrt{2}}{3}\left(\sin\frac{\pi}{2} - \sin\frac{\pi}{4}\right) - \frac{4\sqrt{2}}{3}\left(\cos\frac{\pi}{2} - \cos\frac{\pi}{4}\right)$$

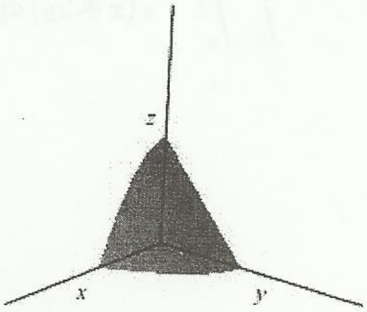
$$= \frac{2\sqrt{2}}{3}\left(1 - \frac{1}{\sqrt{2}}\right) - \frac{4\sqrt{2}}{3}\left(0 - \frac{1}{\sqrt{2}}\right)$$

$$= \frac{2\sqrt{2}}{3} - \frac{2}{3} + \frac{4}{3}$$

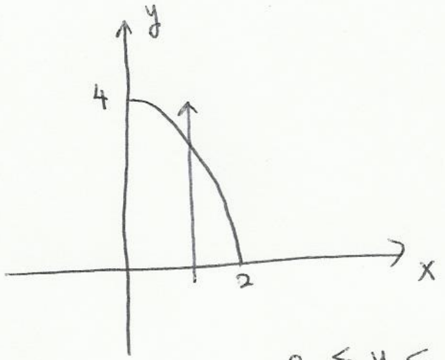
$$= \cancel{\frac{2\sqrt{2}}{3} + \frac{2}{3}} = \cancel{\boxed{\frac{2(\sqrt{2}+2)}{3}}}$$

$$= \frac{2\sqrt{2}}{3} + \frac{2}{3} = \boxed{\frac{2(\sqrt{2}+1)}{\sqrt{3}}}$$

4. Find the volume of the region in the first octant bounded by the coordinate planes and the surface $z = 4 - x^2 - y$. Show your work. (10 points)



$0 \le z \le 4 - x^2 - y$



When $z = 0$

$0 = 4 - x^2 - y$

$y = 4 - x^2$

$0 \le y \le 4 - x^2$

$0 \le x \le 2$

$$\int_0^2 \int_0^{4-x^2} \int_0^{4-x^2-y} dz\,dy\,dx = \int_0^2 \int_0^{4-x^2} (4 - x^2 - y)\,dy\,dx$$

$$= \int_0^2 4y - x^2 y - \frac{y^2}{2} \Big|_0^{4-x^2} dx$$

$$= \int_0^2 4(4-x^2) - x^2(4-x^2) - \frac{(4-x^2)^2}{2}\,dx$$

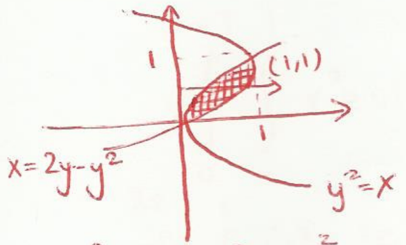
$$= \int_0^2 16 - \cancel{4x^2} - 4x^2 + x^4 - 8 + \cancel{4x^2} - \frac{x^4}{2}\,dx$$

$$= \int_0^2 8 - 4x^2 + \frac{x^4}{2}\,dx = 8x - \frac{4x^3}{3} + \frac{x^5}{10}\Big|_0^2$$

$$= 16 - \frac{32}{3} + \frac{32}{10}$$

$$= \frac{480 - 320 + 96}{30} = \frac{256}{30} = \boxed{\frac{128}{15}}$$

5. Find the moment of inertia (second moment) about the x-axis of a thin plate bounded by the curves $x = y^2$ and $x = 2y - y^2$ if the density at the point (x, y) is $\delta(x, y) = y + 1$. Show your work. (8 points)



$y^2 \le x \le 2y - y^2$

$0 \le y \le 1$

$x = 2y - y^2$

$\Rightarrow x - 1 = -1 + 2y - y^2$

$\Rightarrow x - 1 = -(y-1)^2$

Point of intersection

$y^2 = 2y - y^2$

$2y^2 = 2y$

$y = 0, 1 \Rightarrow x = 0, 1$

$$I_x = \iint_R y^2(y+1)\,dA$$

$$= \int_0^1 \int_{y^2}^{2y-y^2} (y^3 + y^2)\,dx\,dy$$

$$= \int_0^1 y^3x + y^2x \Big|_{y^2}^{2y-y^2} dy$$

$$= \int_0^1 \left[y^3(2y-y^2) + y^2(2y-y^2) - y^5 - y^4\right] dy$$

$$= \int_0^1 \left[2y^4 - y^5 + 2y^3 - y^4 - y^5 - y^4\right] dy$$

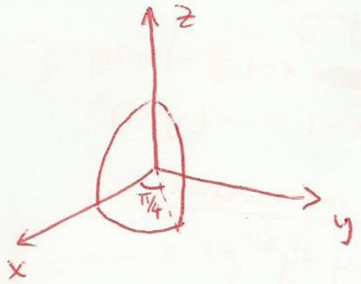
$$= \int_0^1 \left[2y^4 + 2y^3 - 2y^5 - 2y^4\right] dy$$

$$= \cancel{\frac{2y^5}{5}} + \frac{y^4}{2} - \frac{y^6}{3} - \cancel{\frac{2y^5}{5}} \Big|_0^1$$

$$= \cancel{\frac{2}{5} + \frac{2}{3} - \frac{1}{3} - \frac{1}{2} = \frac{2}{5} + \frac{1}{3} - \frac{1}{2} = \frac{2}{5} - \frac{1}{6} = \boxed{\frac{7}{30}}}$$

$$= \frac{1}{2} - \frac{1}{3} = \boxed{\frac{1}{6}}$$

6. Using spherical coordinates, find the volume of the region cut from the solid sphere $\rho \le a$ by the half-planes $\theta = 0$ and $\theta = \pi/4$ in the first octant. Show your work. (8 points)



$$0 \le \rho \le a$$
$$0 \le \phi \le \frac{\pi}{2}$$
$$0 \le \theta \le \frac{\pi}{4}$$

$$\int_0^{\pi/4}\int_0^{\pi/2}\int_0^{a} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{\pi/4}\int_0^{\pi/2} \frac{\rho^3}{3} \sin\phi \Big|_0^a d\phi \, d\theta = \int_0^{\pi/4}\int_0^{\pi/2} \frac{a^3}{3} \sin\phi \, d\phi \, d\theta$$

$$= -\int_0^{\pi/4} \frac{a^3}{3} \cos\phi \Big|_0^{\pi/2} d\theta$$

$$= \frac{a^3}{3}\int_0^{\pi/4} d\theta = \frac{a^3}{3}\cdot\frac{\pi}{4} = \boxed{\frac{\pi a^3}{12}}$$

7. Evaluate the cylindrical coordinate integral. Show your work. (8 points)

$$\int_0^{2\pi}\int_0^1\int_{-1/2}^{1/2} (r^2 \sin^2\theta + z^2)\, dz\, r\, dr\, d\theta$$

$$\int_0^{2\pi}\int_0^1 r^2 z \sin^2\theta + \frac{z^3}{3}\Big|_{-1/2}^{1/2} r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 \left(r^2\sin^2\theta + \frac{1}{12}\right) r\,dr\,d\theta$$

$$= \int_0^{2\pi}\left[\frac{r^4}{4}\sin^2\theta + \frac{r^2}{24}\right]\Big|_0^1 d\theta$$

$$= \int_0^{2\pi}\left[\frac{\sin^2\theta}{4} + \frac{1}{24}\right] d\theta$$

$$= \int_0^{2\pi}\left[\frac{1-\cos 2\theta}{8} + \frac{1}{24}\right] d\theta$$

$$= \frac{\theta}{8}\Big|_0^{2\pi} + \frac{\theta}{24}\Big|_0^{2\pi} - \frac{\sin 2\theta}{16}\Big|_0^{2\pi}$$

$$= \frac{\pi}{4} + \frac{\pi}{12} = \frac{4\pi}{12} = \boxed{\frac{\pi}{3}}$$

8. Evaluate

$$\int_C (xy + y + z)\, ds$$

along the curve

$$\mathbf{r}(t) = 2t\mathbf{i} + t\mathbf{j} + (2 - 2t)\mathbf{k}, \quad 0 \le t \le 1.$$

Show your work. (8 points)

$$r'(t) = 2\hat{i} + \hat{j} - 2\hat{k}$$

$$|r'(t)| = \sqrt{2^2 + 1^2 + (-2)^2} = \sqrt{9} = 3$$

$$\int_C (xy + y + z)\, ds = \int_0^1 (2t^2 + t + 2 - 2t) \cdot 3\, dt$$

$$= \int_0^1 (2t^2 - t + 2)\, 3\, dt$$

$$= 3\left[\frac{2t^3}{3} - \frac{t^2}{2} + 2t\right]\Big|_0^1$$

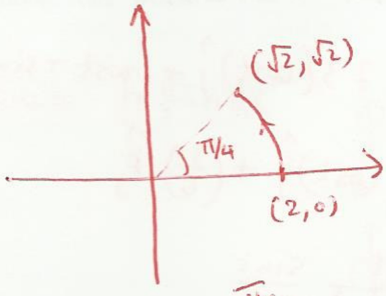
$$= 3\left[\frac{2}{3} - \frac{1}{2} + 2\right]$$

$$= 2 - \frac{3}{2} + 6 = 8 - \frac{3}{2} = \boxed{\frac{13}{2}}$$

9. Evaluate

$$\int_C (x+y)\,ds$$

where $C : x^2 + y^2 = 4$ in the first quadrant from (2, 0) to $(\sqrt{2}, \sqrt{2})$. Show your work. (8 points)



$r(t) = (2\cos t)\,\hat{i} + (2\sin t)\,\hat{j}$

$0 \le t \le \frac{\pi}{4}$

$r'(t) = (-2\sin t)\,\hat{i} + (2\cos t)\,\hat{j}$

$|r'(t)| = \sqrt{4\sin^2 t + 4\cos^2 t} = 2.$

$$\int_C (x+y)\,ds = \int_0^{\pi/4} (2\cos t + 2\sin t) \cdot 2\,dt$$

$$= 4\left[\sin t - \cos t\right]\Big|_0^{\pi/4}$$

$$= 4\cdot\left[\cancel{\sin \pi/4} - \cancel{\cos \pi/4} - \sin 0 + \cos 0\right]$$

$$= 4\,[1] = \boxed{4}$$

10. Find the work done by **F** over the curve in the direction of increasing t. Show your work.

(8 points)

$$\mathbf{F} = 2y\mathbf{i} + 3x\mathbf{j} + (x+y)\mathbf{k}$$

$$\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + (t/6)\mathbf{k}, \quad 0 \le t \le 2\pi$$

$$W = \int_C F \cdot dr, \quad F(r(t)) = (2\sin t)\hat{i} + 3(\cos t)\hat{j} + (\cos t + \sin t)\hat{k}$$

$$\frac{dr}{dt} = (-\sin t)\hat{i} + (\cos t)\hat{j} + \left(\frac{1}{6}\right)\hat{k}$$

$$F(r(t)) \cdot \frac{dr}{dt} = -2\sin^2 t + 3\cos^2 t + \frac{\cos t}{6} + \frac{\sin t}{6}$$

$$= 2\cos 2t + \cos^2 t + \frac{\cos t}{6} + \frac{\sin t}{6}$$

$$W = \int_0^{2\pi} \left[2\cos 2t + \cos^2 t + \frac{\cos t}{6} + \frac{\sin t}{6}\right] dt$$

$$= \int_0^{2\pi} \left[2\cos 2t + \frac{1+\cos 2t}{2} + \frac{\cos t}{6} + \frac{\sin t}{6}\right] dt$$

$$= \sin 2t \Big|_0^{2\pi} + \frac{t}{2}\Big|_0^{2\pi} + \frac{\sin 2t}{4}\Big|_0^{2\pi} + \frac{\sin t}{6}\Big|_0^{2\pi} - \frac{\cos t}{6}\Big|_0^{2\pi}$$

$$= \pi - \left(\frac{\cos 2\pi}{6} - \frac{\cos 0}{6}\right)$$

$$= \boxed{\pi}$$

11. Find the flux of the field

$$\mathbf{F} = 2x\mathbf{i} - 3y\mathbf{j}$$

across the circle

$$\mathbf{r}(t) = (a\cos t)\mathbf{i} + (a\sin t)\mathbf{j}.$$

Show your work. (8 points)

$$\text{Flux} = \oint_C \mathbf{F}\cdot\mathbf{n}\,ds = \oint_C (M\,dy - N\,dx) \qquad M = 2x \quad N = -3y.$$

$$x = a\cos t \qquad dx = -a\sin t\,dt$$

$$y = a\sin t \qquad dy = a\cos t\,dt$$

$$= \oint_C (2x\,dy + 3y\,dx)$$

$$= \int_0^{2\pi} \left[2a\cos t\, a\cos t\,dt + 3a\sin t\,(-a\sin t)\,dt\right]$$

$$= \int_0^{2\pi} 2a^2\cos^2 t\,dt - \int_0^{2\pi} 3a^2\sin^2 t\,dt$$

$$= \int_0^{2\pi} 2a^2\left(\frac{1+\cos 2t}{2}\right)dt - \int_0^{2\pi} 3a^2\left(\frac{1-\cos 2t}{2}\right)dt$$

$$= a^2 t\Big|_0^{2\pi} + \frac{a^2\sin 2t}{2}\Big|_0^{2\pi} - \frac{3a^2}{2}t\Big|_0^{2\pi} + \frac{3a^2\sin 2t}{4}\Big|_0^{2\pi}$$

$$= 2\pi a^2 - 3\pi a^2$$

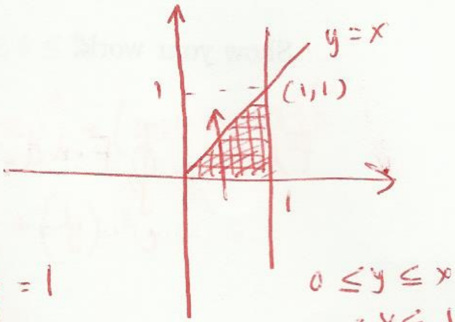
$$= \boxed{-\pi a^2}$$

12. Use Green's Theorem to find the counterclockwise circulation and outward flux for the field $\mathbf{F}$ and curve C. Show your work. (10 points)

$$\mathbf{F} = (x+y)\mathbf{i} - (x^2+y^2)\mathbf{j}$$

C: The triangle bounded by $y=0, x=1$, and $y=x$.

$M = x+y \qquad N = -(x^2+y^2)$



$0 \le y \le x$

$0 \le x \le 1$

Circulation

$$\oint_C \mathbf{F}\cdot\mathbf{T}\,ds = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$$

$\frac{\partial M}{\partial y} = 1$

$\frac{\partial N}{\partial x} = -2x$

$$= \int_0^1\int_0^x (-2x-1)\,dy\,dx$$

$$= \int_0^1 -2xy - y\Big|_0^x dx = \int_0^1 (-2x^2-x)\,dx = -\frac{2x^3}{3} - \frac{x^2}{2}\Big|_0^1 = -\frac{2}{3} - \frac{1}{2}$$

$$= \boxed{-\frac{7}{6}}$$

Flux

$$\oint_C \mathbf{F}\cdot\mathbf{n}\,ds = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dA$$

$\frac{\partial M}{\partial x} = 1$

$\frac{\partial N}{\partial y} = -2y$

$$= \int_0^1\int_0^x (1-2y)\,dy\,dx$$

$$= \int_0^1 y - y^2\Big|_0^x dx = \int_0^1 (x-x^2)\,dx = \frac{x^2}{2} - \frac{x^3}{3}\Big|_0^1$$

$$= \frac{1}{2} - \frac{1}{3}$$

$$= \boxed{\frac{1}{6}}$$