SOLUTIONS

Math 1260 Final Exam - June 22, 2006

1. **(15 points)** The element fermium has a half-life of 20 hours.
   a. How much of a sample of fermium weighing 50 grams will remain after 48 hours?
   b. How much time is necessary until the sample is 95% gone? (so only 5% of the original amount remains).

$y = y_0 e^{kt}$

$\frac{1}{2} y_0 = y_0 e^{20k}$

$\ln(\frac{1}{2}) = 20k$

$k \to -.03466$

$y = 50 e^{-.03466t}$

a. plug in $t = 48$

$y = 50 e^{-.03466 \cdot 48} = \boxed{9.47 \text{ grams}}$

b. Set $y = .05 \cdot 50$

$2.5 = 50 e^{-.03466t}$

$.05 = e^{-.03466t}$

$t = \frac{\ln(.05)}{-.03466} = \boxed{86.43 \text{ hours}}$

2. **(16 points)** Solve for x. Round your answer to the nearest 100th.

   a. $5^x = 9$.

$x \ln 5 = \ln 9 \qquad x = \frac{\ln 9}{\ln 5} = \boxed{1.37}$

   b. $3\log_4(3x) = 9$

$\log_4 3x = 3 \qquad 4^3 = 3x \qquad 64 = 3x \qquad x = 64/3 = \boxed{21.33}$

3. **(12 points)** Suppose the demand for a certain item is given by $D(p) = -2p^2 - 4p + 300$ where p represents the price of the item in dollars. When the price is 10, find the rate of change of demand with respect to price.

$D'(p) = -4p - 4$

$D'(10) = \boxed{-44 \text{ units}/\$}$

4. **(40 points)** For each function $f(x)$ below, calculate the derivative $f'(x)$:

a. $f(x) = (3x+9)^5$

$$\boxed{5(3x+9)^4 \cdot 3}$$

b. $f(x) = e^{-3x^2}$

$$\boxed{-6x \cdot e^{-3x^2}}$$

c. $f(x) = \frac{2}{x^5} + \frac{3}{x^3}$

$$\boxed{-10x^{-6} - 9x^{-4}}$$

d. $f(x) = \ln(x)$

$$\boxed{\frac{1}{x}}$$

e. $f(x) = (\sqrt{x} + x^2)(\ln(3x+5))$

$$\boxed{\left(\frac{1}{2\sqrt{x}} + 2x\right)(\ln(3x+5)) + (\sqrt{x} + x^2) \cdot \frac{3}{3x+5}}$$

f. $f(x) = 7^x$

$$\boxed{7^x \ln 7}$$

g. $f(x) = 6x^5 - 4x^3 + 12x^2 - 19$

$$30x^4 - 12x^2 + 24x$$

h. $f(x) = (\sqrt{2} + \sqrt[4]{17})^{36-15e^3}\sqrt[7]{\pi}$

$$0$$

i. $f(x) = 10$

$$0$$

j. $f(x) = \frac{x^2e^x}{x-5}$

$$\frac{(x-5)(2xe^x + x^2e^x) - x^2e^x}{(x-5)^2}$$

5. **(12 points)** The position of a car at time t hours is given in miles by: $p(t) = t^2 + 40t + 10$.
a. Find the average speed of the car (in miles/hour) from $t = 3$ to $t = 6$.
b. Find the time between $t = 3$ and $t = 6$ when the car's instantaneous speed was exactly this average speed.

a. $\frac{p(6) - p(3)}{6-3} = \frac{286 - 139}{3} = \frac{147}{3} =$ 49 m/hour

b. $p'(t) = 2t + 40$

$2t + 40 = 49$

$t = 4.5$ hours

6. **(12 points)**

a. Give the definition of the derivative of a function $f(x)$.

b. *Using the definition*, show that the derivative of $f(x) = x^2$ is $2x$.

a. $f'(x) = \lim\limits_{h \to 0} \frac{f(x+h) - f(x)}{h}$

b. $\lim\limits_{h \to 0} \frac{(x+h)^2 - x^2}{h} = \lim\limits_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h}$

$= \lim\limits_{h \to 0} \frac{\cancel{h}(2x+h)}{\cancel{h}}$

$= \lim\limits_{h \to 0} 2x+h = \boxed{2x}$

7. **(10 points)**

a. Find the equation of the tangent line to the graph $y = x^2 + 3x - 2$ at the point where $x = -2$.

b. Find the x value(s) where the graph has a horizontal tangent line.

a. $\frac{dy}{dx} = 2x + 3$ Slope $= 2(-2)+3 = -1$

point $= (-2, -4)$

$\boxed{y + 4 = -1(x+2)}$

b. $2x+3=0$

$\boxed{x = -3/2}$

8. **(8 points)** Using calculus, explain why the the vertex of the parabola $y = ax^2 + bx + c$ occurs where $x = -b/2a$.

The vertex occurs when the tangent line is horizontal, i.e. when $\frac{dy}{dx} = 0$.

$\frac{dy}{dx} = 2ax + b$ which is 0 exactly when $x = -b/2a$.

9. **(15 points)** Evaluate the following limits. If the limit does not exist then write DNE.

a. $\lim_{x \to 3} e^x$

e^3

b. $\lim_{x \to 0} \frac{|x|}{x}$

DNE

c. $\lim_{x \to -\infty} \frac{6x^3 + 2x^2 - \sqrt{x}}{7x^3 - 5x^2 + 9}$

$6/7$

d. $\lim_{x \to 2} \frac{x+2}{x-2}$

DNE

e. $\lim_{x \to 2} \frac{x^2 - 4}{x-2}$ $= \lim_{x \to 2} \frac{(x+2)\cancel{(x-2)}}{\cancel{x-2}} = 4$

10. **(20 points)** Use the graph of $y = f(x)$ to answer the following. If a limit does not exist then write DNE.

a. $\lim_{x\to 3^+} f(x)$

4

b. $\lim_{x\to 3^-} f(x)$

-2

c. $f(3)$

1

d. $\lim_{x\to -2} f(x)$

DNE

e. $\lim_{x\to 0} f(x)$

1

f. $\lim_{x\to \infty} f(x)$

1

g. $\lim_{x\to -\infty} f(x)$

-3

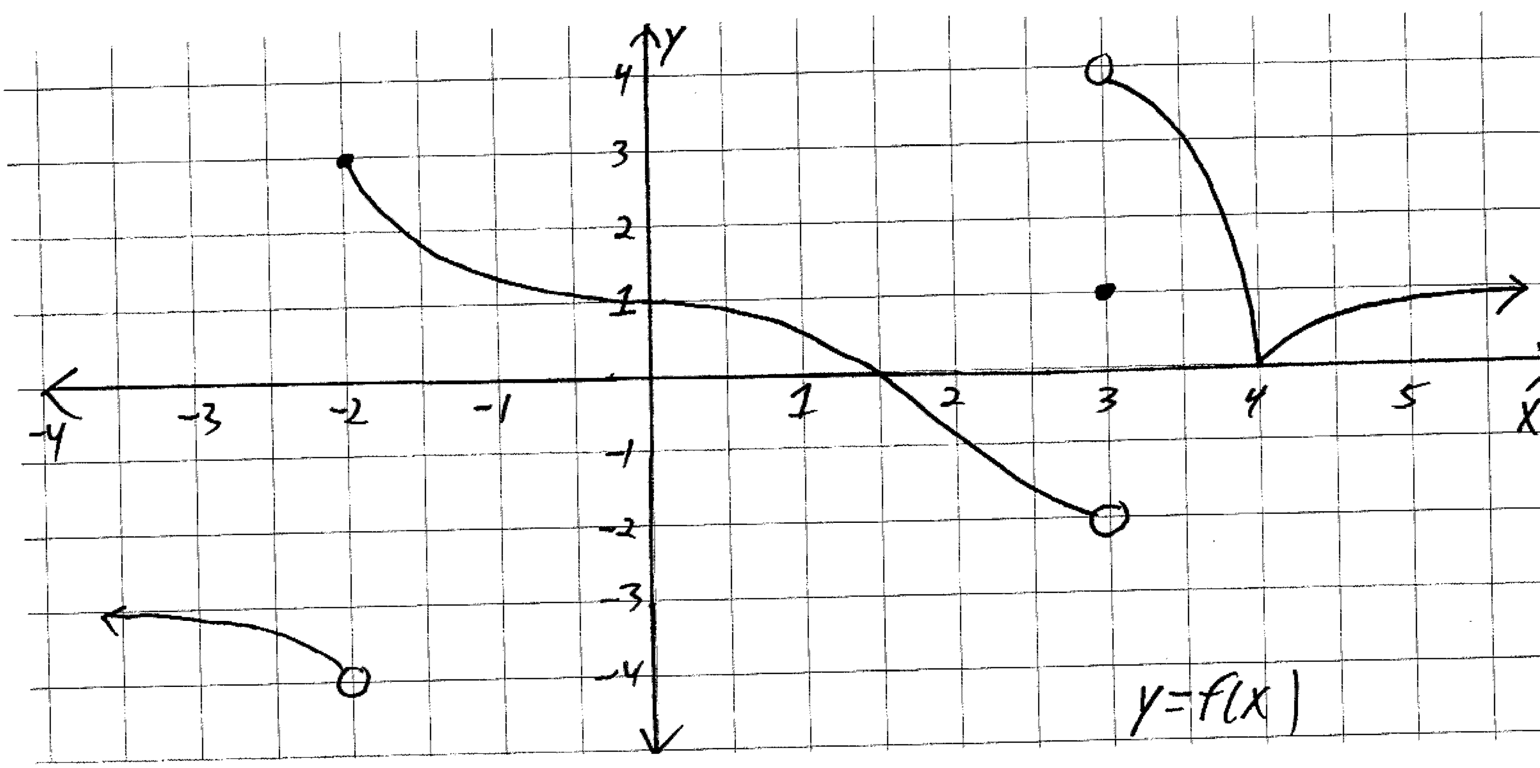
h. Is $f(x)$ differentiable at $x = 4$?

No

i. Is $f(x)$ continuous at $x = 4$?

yes

j. What, if any, horizontal asymptotes does $f(x)$ have? $y = 1, y = -3$



11. **(10 points)** The function $f(x)$ is *continuous at* $x{=}c$ if...

1. $\lim_{x \to c} f(x)$ exists

2. $f(c)$ is defined

3. 1 & 2 are =

12. **(10 points)** Suppose $f(x)$ and $g(x)$ are differentiable and consider the following data:

$f(2) = -1,\ f'(2) = -3,\ f(6) = 1,\ f'(6) = -6,\ g(2) = 4,\ g'(2) = -3,\ g(6) = 2,\ g'(6) = 5.$

Let $h(x) = f(g(x))$. Use the chain rule to find $h'(6)$. Hint: You don't need to use all the information above!

$h'(6) = f'(g(6)) \cdot g'(6) = f'(2) \cdot g'(6) = (-3)(5) = -15$

13. **(20 points)** A ball is thrown upward from a 64 foot building. The equation for its height at time t seconds is:

$$h(t) = -16t^2 + 8t + 64.$$

a. At what time does the ball hit the ground?
b. Find an equation for the velocity of the ball at time t.
c. At what initial ($t = 0$) velocity was the ball thrown.
d. How fast is the ball going when it hits the ground?
e. What is the ball's maximum height?

a. $-16t^2 + 8t + 64 = 0$

$2t^2 - t - 8 = 0 \qquad t = \frac{1 \pm \sqrt{65}}{4}$

$t = \frac{1 + \sqrt{65}}{4} \approx$ 2.27 seconds

b. $V(t) = h'(t) = \boxed{-32t + 8}$

c. $V(0) = \boxed{8 \text{ ft/sec}}$

d. $V(2.27) = -32(2.27) + 8 = -64.64$

$\boxed{64.64 \text{ ft/second}}$ down

e. vertex is $t = \frac{-8}{-32} = 1/4$

$h(1/4) = -16 \cdot \left(\frac{1}{4}^2\right) + 8\left(\frac{1}{4}\right) + 64 = -1 + 2 + 64 ...$