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10/12/06

Recall $\frac{dy}{dx}$ is instantaneous rate of change of y with respect to x.

Units of $\frac{dy}{dx}$ are units of y / units of x.

Problem (example 1)

Air is pumped into spherical balloon at $100\,cm^3/s$. How fast is radius of balloon increasing when diameter is 50 cm.

What is given? * $\frac{dV}{dt} = 100\,cm^3/s$ (air pumped in is rate of change of volume w/r time).

What do we want? $\frac{dr}{dt}$ when $r = 25\,cm$.

Notice that V and r are both functions of t, since both change over time.

Procedure 1. ~~~~ Find equation which relates V & r.

$$V = \frac{4}{3}\pi r^3 \quad \text{think: } V(t) = \frac{4}{3}\pi r(t)^3$$

2. Differentiate both sides implicitly with respect to t.

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

* this step always uses chain rule *

3. Plug in into

$$100\,cm^3/s = 4\pi (25\,cm)^2 \frac{dr}{dt}$$

$$100\,cm^3/s = 2500\,cm^2 \pi \frac{dr}{dt}$$

$$\boxed{\frac{dr}{dt} = \frac{1}{25\pi}\,cm/sec}$$

Example #2 (Famous ladder problem!!)

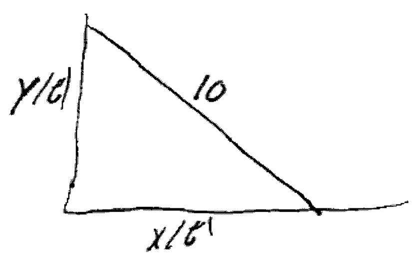
A ladder 10 ft long rests against vertical wall. The bottom slips away at 1 ft/s. How fast is the top sliding down when the bottom of the ladder is 6 ft from the wall?

Bonus: What happens to the velocity of the top as the bottom approaches 10 ft from wall.

Solution

Step 1: Draw a picture

Step 2: Assign names to all quantities which vary over time.



x = distance from base of ladder to wall

y = height of top of ladder

Step 3 Find equation relating the quantities.
Also express given information in "terms of calculus"

$$x(t)^2 + y(t)^2 = 100$$

Given: $\frac{dx}{dt} = 1$ ft/sec

Find $\frac{dy}{dt}$ @ when $x = 6$

Step 4 Diff w.r.t. t

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

When $x=6$ then $y=8$.

$$12 \cdot 1\text{ ft/sec} + 16 \cdot \frac{dy}{dt} = 0$$

$$\frac{dy}{dt} = -.75 \text{ ft/sec}$$

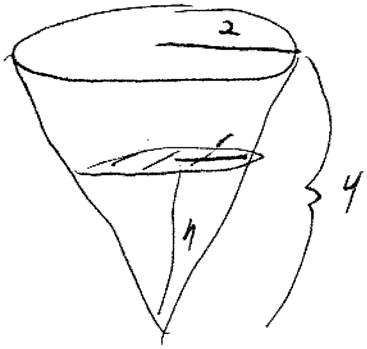
Bonus $2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0$ Given $\frac{dx}{dt} = 1$.

Thus $2x + 2y \frac{dy}{dt} = 0$

$$\frac{dy}{dt} = \frac{-x}{y}$$

Thus as $x \to 10$ and $y \to 0$ $\frac{dy}{dt} \to -\infty$, the tip moves infinitely fast! (Actually in real life it separates from the wall)

Example 3 Given circular cone water tank, radius 2m, height 4m



Water pumped in at $2m^3/min$

Find rate water level is rising when water is 3ft down

Let $h = h(t)$ be height of water

$r = r(t)$ radius as above

$V = \frac{1}{3}\pi r^2 h$. Given $\frac{dV}{dt} = 2m^3/min$ Find $\frac{dh}{dt}$ when $h = 3ft$.

Seems to be an extra variable r. However $\frac{r}{h} = \frac{2}{4}$ (similar Δ)

so $r = \frac{h}{2}$ Thus

$V = \frac{\pi}{12} h^3$ $\quad \frac{dV}{dt} = \frac{\pi}{4} h^2 \frac{dh}{dt}$

$2m^3/min = \pi/4 \cdot 3^2 \frac{dh}{dt}$

$$\boxed{\frac{dh}{dt} = \frac{8}{9\pi} \, m/min}$$

General procedure

1. Read problem and (usually) draw a picture.

2. Give names to all quantities which are functions of time

3. Express given and desired quantities (usually rates of change)

4. Write equation relating various quantities
   - geometry of situation
   - formulas for areas, volumes, etc.
   - perhaps eliminate variables

5. Apply $\frac{d}{dt}$ to each side using chain rule

6. Substitute & solve

Warning: Do not sub in specific values for anything that changes over time until after you take derivative with respect to time.

Problems

10, 11, 16*