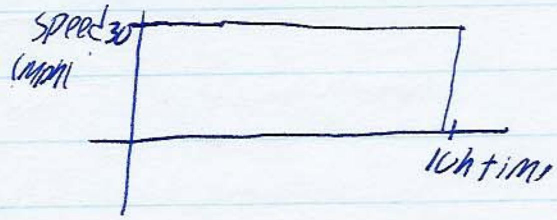
11/16/06

Consider:



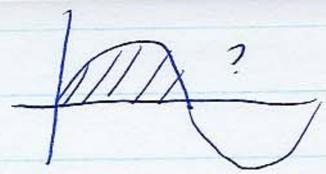
Q: How far did car travel? Given graph of $x'(t)$ what is total $x(t)$?

Total is time · speed = area under curve.

Question 1. What is area of a rectangle?

2. What is area of a circle?

3. What is area under graph of $y = \sin x$ from 0 to π



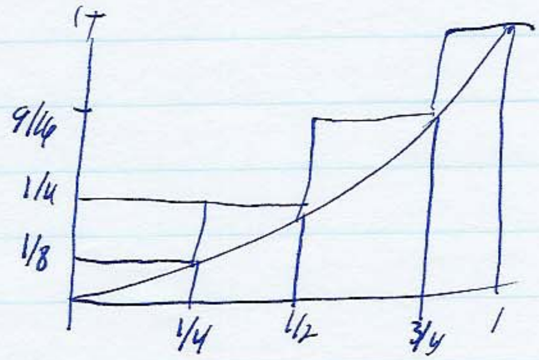
Procedure Similar to finding ~~eq~~ slope of tangent line!

1. Estimate area under curve by rectangles

2. Let the rectangles get smaller & smaller (and more numerous).

3. Take a limit!

<u>Problem</u> Find area under $y=x^2$ from $x=0$ to 1



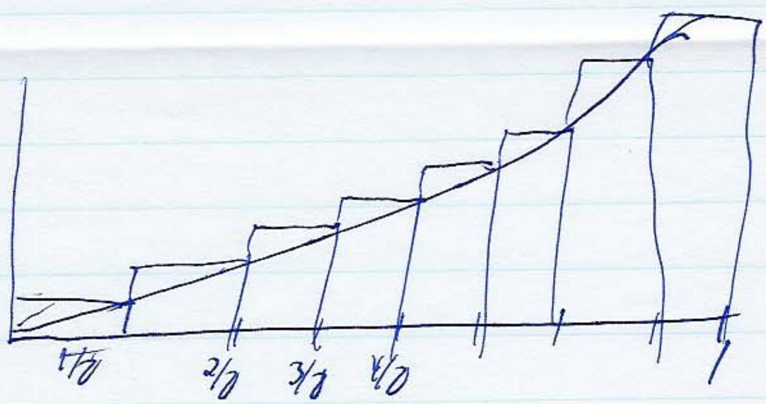
4 rectangles.

$$\text{Area} = \tfrac{1}{4}\cdot\left(\tfrac{1}{4}\right)^2 + \tfrac{1}{4}\left(\tfrac{1}{2}\right)^2 + \tfrac{1}{4}\left(\tfrac{3}{4}\right)^2 + \tfrac{1}{4}(1)^2$$

$$= \tfrac{1}{4}\left(\tfrac{1}{4}^2 + \tfrac{1}{2}^2 + \tfrac{3}{4}^2 + 1^2\right)$$

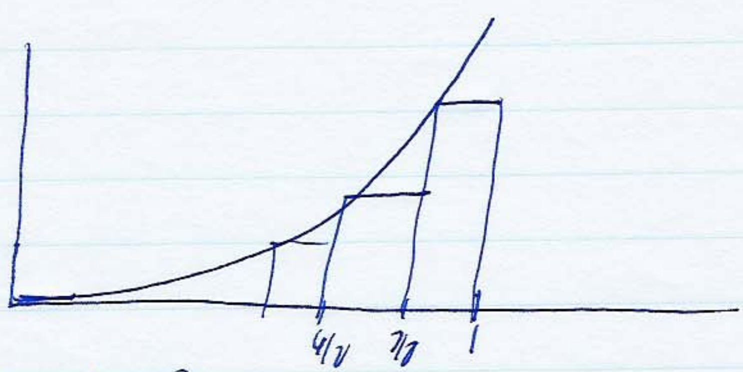
$$= \tfrac{1}{4}\left(\tfrac{30}{16}\right) = \tfrac{15}{32}$$ Definitely overestimate

8 rectangles.



$$\text{Area} = \tfrac{1}{8}\left(\tfrac{1}{8}^2 + \tfrac{2}{8}^2 + \tfrac{3}{8}^2 + \tfrac{4}{8}^2 + \dots + 1^2\right) = .3984$$ overestimate

8 small rectangles



$$\text{Area} = \tfrac{1}{8}\left(0^2 + \tfrac{1}{8}^2 + \tfrac{2}{8}^2 + \dots + \tfrac{7}{8}^2\right) = .2734$$

n rectangles, equal base, overestimate

$$\text{Area} = \frac{1}{n}\left[\left(\frac{1}{n}\right)^2 + \left(\frac{2}{n}\right)^2 + \dots + \left(\frac{n}{n}\right)^2\right]$$

$$= \frac{1}{n^3}\left(1^2 + 2^2 + \dots + n^2\right)$$

Thm $1^2 + 2^2 + \dots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$

$$\text{Area} = \frac{1}{n^3} \cdot \cancel{2n^3+} = \frac{1}{n^2} \cdot \frac{2n^2+3n+1}{6} = \frac{2 + \frac{3}{n} + \frac{1}{n^2}}{6}$$

so $\lim\limits_{n\to\infty} \text{Area} = 1/3$ Thus

Area $\le 1/3$

n rectangles, underestimate

$$\text{Area} = \frac{1}{n}\left(\frac{0}{n^2} + \frac{1}{n^2} + \dots + \left(\frac{n-1}{n}\right)^2\right)$$

$$= \frac{1}{n^3} \cdot \frac{(n-1)n(2n-1)}{6} = \frac{2n^2-3n+1}{6n^2} = \frac{2 - 3/n + 1/n^2}{6}$$

$\lim\limits_{n\to\infty} = 1/3$ so area $\ge 1/3$

** Area under $y = \frac{1}{x^2}$ from $x=0$ to $x=1$

is exactly $1/3$ **