Math 1830 Project Number 3- Why astronomers use parabolic telescopes.

Directions: This project is due at the beginning of class on Wednesday October 18. Do not wait until the last minute to work on this project!!

Introduction: The star nearest our sun is Alpha Centauri, which is about 4 light years from earth. It is so far away that when its light reaches earth, it is travelling in essentially parallel rays. To observe distant stars, astronomers use mirrors shaped like paraboloids, which are parabolas rotated about their axes. The reason they use a paraboloidal mirror is that it focuses all the light to a single point, the "focus." In this project you will demonstrate the focusing properties of parabolas.

Problems:

a. Suppose our mirror is shaped like the parabola $y = cx^2$, where c is any positive constant. Find the coordinates of its focus and the equation of its directrix in terms of c. (We are going to assume our mirror is just in the xy-plane)

b. Find the equation of the tangent line to the parabola at an arbitrary point (x_0, y_0) on the parabola. (so $y_0 = cx_0^2$) Then, find the y-intercept of the tangent line.

c. Consider the triangle formed by the point (x_0, y_0) on the parabola, the y-intercept of the tangent line at this point, and the focus. Draw a picture of this triangle. Prove the triangle is isosceles.

d. Suppose an incoming light ray strikes a curve at a point (x_0, y_0). If the light ray makes an angle α with respect to the tangent line, then it is reflected at an equal angle to the tangent line. This result from physics is known by the phrase "the angle of incidence equals the angle of reflection." Using this fact, argue that incoming light rays parallel to the axis of the parabola are all reflected to the focus, independent of the point of incidence. Thus a parabolic mirror focuses incoming light rays parallel to the axis to a point.

e. The path followed by a ray of light from the star to the focus of the mirror has another special property. Draw a chord of the parabola that is above the focus and parallel to the directrix. Consider a ray of light parallel to the axis as it crosses the chord, hits the parabola and is reflected to the focus. Let d_1 be the distance from the chord to the point of incidence (x_0, y_0) on the parabola and let d_2 be the distance from (x_0, y_0) to the focus. Show that the sum of the distances $d_1 + d_2$ is constant, independent of the particular point of incidence. (This property is important. It guarantees that light rays which leaves the star at the same time arrive at the focus at the same time.)

f. Draw and clearly label a diagram illustrating what you have shown in the project.

Extra credit: Now suppose the mirror really is a paraboloid, i.e. the curve $y = cx^2$ rotated around the y-axis. Explain why all incident rays parallel to the axis of such a mirror focus to a point. Also prove that light travelling along different rays to the focus from a chord perpendicular to the axis will have travelled the same distance, even though these rays were reflected from different points of incidence on the mirror.

∗Project idea from "Student research projects in calculus" published by the Mathematical Association of America.