Upper and Lower Riemann Sums

Let $f(x)$ be a positive, continuous, increasing function on $[a, b]$. Your goal is to develop a technique to estimate the integral $\int_a^b f(x)dx$ within a given margin of error without evaluating the integral.

a. Your first step is to understand two special kinds of Riemann sums. Partition $[a, b]$ into n equal parts. What is the length of each subinterval? For the first kind of Riemann sum, choose the height of each rectangle to be the maximum value of f over the subinterval. How do you know the maximum exists? For what value of x do you get the maximum value? The Riemann sum defined by taking the maximum value of f over each subinterval is called an *upper sum.* Explain why $\int_a^b f(x)dx$ is less than or equal to any upper sum. Write explicitly the upper sum when $n = 4$.

b. For the second kind of Riemann sum, repeat part (a), but replace the maximum value of f over each subinterval by the minimum value. This Riemann sum is called a *lower sum.* Explain why $\int_a^b f(x)dx$ is greater than or equal to any lower sum. Write explicitly the lower sum when $n = 4$.

c. Let U_4 be the upper sum and L_4 the lower sum for $n = 4$ found above. The difference $U_4 - L_4$ is nonnegative. Why? Simplify $U_4 - L_4$ as much as possible. Next, find a very simple expression for $U_n - L_n$. Explain why your expression is correct.

d. Part (c) gives you a way to make the upper and lower sums as close to each other as you wish. Suppose ϵ is a given positive real number. How can you use the result of part (c) to estimate $\int_a^b f(x)dx$ with a margin of error at most ϵ? Explain.

e. Estimate $\int_{0.5}^{2}(1 + x^2)^{1/3}dx$ with an error at most 10^{-1} using the ideas from part (d).

Extra credit: Can you think of other ways (other than the maximum or minimum) to choose the value of f on each subinterval that would give better approximations to the integral than U_n or L_n? Discuss your ideas.