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6. $a_n = (-1)^n \cdot \frac{n}{(n+1)^2}$

7. $a_n = 5n - 3$

9. $\lim_{n\to\infty} \frac{3+5n^2}{n+n^2} = \lim_{n\to\infty} \frac{3/n^2+5}{1/n+1} = 5$

13. $a_n = \frac{(n+2)!}{n!} = (n+2)(n+1)$ This diverges.

15. $\lim_{n\to\infty} \frac{(-1)^{n-1} n}{n^2+1} = 0$

22. $\{n\cos \pi n\} = \cos\pi, 2\cos 2\pi, 3\cos 3\pi, \ldots = -1, 2, -3, 4, \ldots$ diverges

25. diverges

29. $a_1 = 1060$, $a_2 = 1123.60$, $a_3 = 1191.02$, $a_4 = 1262.48$

$a_5 = 1338.23$

b. This diverges. Notice $(1.06)^{13} = 2.13 > 2$ Thus

$a_{13} > 2000$

$a_{20} > 4000$ etc...

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24. $a_n = \frac{\sin 2n}{1+\sqrt{n}}$ since $\frac{-1}{1+\sqrt{n}} \le a_n \le \frac{1}{1+\sqrt{n}}$

then $\lim_{n\to\infty} a_n = 0$ by squeeze thm.

28. $a_n = \frac{(-3)^n}{n!}$ Let's see that $\lim_{n\to\infty} |a_n| = 0$ so $\lim_{n\to\infty} a_n = 0$

$\frac{3}{1} = |a_1|$, $|a_2| = \frac{3 \cdot 3}{2 \cdot 1}$ $|a_3| = \frac{3 \cdot 3 \cdot 3}{3 \cdot 2 \cdot 1}$ $|a_4| = \frac{3 \cdot 3 \cdot 3 \cdot 3}{4 \cdot 3 \cdot 2 \cdot 1}$

Notice $|a_n| = \underbrace{\frac{3}{n} \cdot \frac{3}{n-1} \cdot \ldots \cdot \frac{3}{3}}_{<1} \cdot \frac{3 \cdot 3}{2 \cdot 1}$

Thus $0 < |a_n| < \frac{27}{2} \cdot \frac{1}{n}$

But $\lim_{n\to\infty} \frac{27}{2} \cdot \frac{1}{n} = \frac{27}{2} \lim_{n\to\infty} \frac{1}{n} = 0$ so $\lim_{n\to\infty} |a_n| = 0$

by Squeeze Thm.

31. Given that $\{a_n\}$ is monotonic decreasing and bounded it must converge.
The limit will lie in $[5, 8]$

46. a. Suppose $\lim_{n\to\infty} a_{2n} = L$, $\lim_{n\to\infty} a_{2n+1} = L$.

Let $\varepsilon > 0$ be given. By the first limit we know there is $\tilde{N} > 0$ so that $2n > \tilde{N} \Rightarrow |a_{2n} - L| < \varepsilon$. By the second we know there is $M > 0$ so $2n+1 > M \Rightarrow |a_{2n+1} - L| < \varepsilon$.

So if we choose $N > \max(\tilde{N}, M)$ then any $n > N$ means $n > \tilde{N}$, $n > M$ so $|a_n - L| < \varepsilon$. Thus $\lim_{n\to\infty} a_n = L$.

b. $a_1 = 1$ $\quad a_2 = 1 + \frac{1}{1+1} = \frac{3}{2}$

$a_3 = 1 + \frac{1}{1+a_2} = 1 + \frac{1}{5/2} = 7/5$

$a_4 = 1 + \frac{1}{1+a_3} = 1 + \frac{1}{1+7/5} = 19/12$

$a_5 = 1 + \frac{1}{1+a_4} = 1 + \frac{1}{1+19/12} = 43/31$

$a_6 = 1 + \frac{1}{1+a_5} = 1 + \frac{1}{1+43/31} = 1 + \frac{31}{74} = \frac{105}{74}$

$a_7 = 1 + \frac{1}{1+\frac{105}{74}} = 1 + \frac{1}{\frac{179}{74}} = 1 + \frac{74}{179} = \frac{253}{179}$

$a_8 = 1 + \frac{1}{1+\frac{253}{179}} = 1 + \frac{1}{\frac{432}{179}} = 1 + \frac{179}{432} = \frac{611}{432}$

Notice $\{a_{2n}\}$ and $\{a_{2n+1}\}$ are both ~~increasing~~ decreasing and bounded ~~above~~ below by $1 + \frac{1}{1+2} = 1\frac{1}{3}$

so they converge.

44. Prove $\lim_{n\to\infty} r^n = 0$ if $|r|<1$.

Rmk If we can prove $\lim_{n\to\infty} |r|^n = 0$ we are done by Thm 6. So assume $r>0$ without loss of generality.

Now let $\varepsilon>0$ be given. If $\varepsilon>1$ then $N=1$ works since $|r^n-0|<\varepsilon$ for all n.

So assume $\varepsilon<1$. Chose $N > \frac{\ln\varepsilon}{\ln r}$. Then

Suppose $n>N$ so

$$n > \frac{\ln\varepsilon}{\ln r}$$

$$\ln r \, n < \ln\varepsilon \quad \text{since } \ln r<0 \text{ because } r<1$$

$$(e^{\ln r})^n < e^{\ln\varepsilon}$$

$r^n < \varepsilon$, as desired.

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3. $5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27}$ $a = 5$ $r = -2/3$

series converges, sum is $\frac{5}{1 - -2/3} = \frac{5}{5/3} = 3$

4. $1 + .4 + .16 + .064$

$a = 1,\ r = .4$

converges, sum is $\frac{1}{1-.4} = \frac{1}{.6} = 5/3$

5. $\sum_{n=1}^{\infty} 5 \cdot \left(\frac{2}{3}\right)^{n-1}$ $a = 5,\ r = 2/3$

converges to $\frac{5}{1-2/3} = 15$

6. $\sum_{n=1}^{\infty} \frac{(-6)^{n-1}}{5^{n-1}} = 1 - \frac{6}{5} + \frac{36}{25} \ldots$

diverges! $r = -6/5$

45. Since all the a_n are positive then

$$S_n = a_1 + a_2 + \ldots + a_n > a_1 + a_2 + \ldots + a_{n-1} = S_{n-1}$$

Thus $S_{n-1} < S_n$ so $\{S_n\}$ is monotonic increasing and bounded above by 1000. Thus $\{S_n\}$ converges so Σa_n converges.