Project #3 Suggested Solution

a. The focus of $x^2 = 4ay$ is $(0, a)$ and directrix is $y = -a$.
Our parabola is

$$x^2 = 4 \cdot \frac{1}{4c} y^2 \text{ so}$$

$$\boxed{\begin{array}{l} \text{Focus } (0, 1/4c) \\ \text{Directrix } \quad y = -1/4c \end{array}}$$

b. $y = cx^2 \quad y' = 2cx$ thus at an arbitrary point (x_0, cx_0^2) the slope of the tangent is $2cx_0$.

Equation:

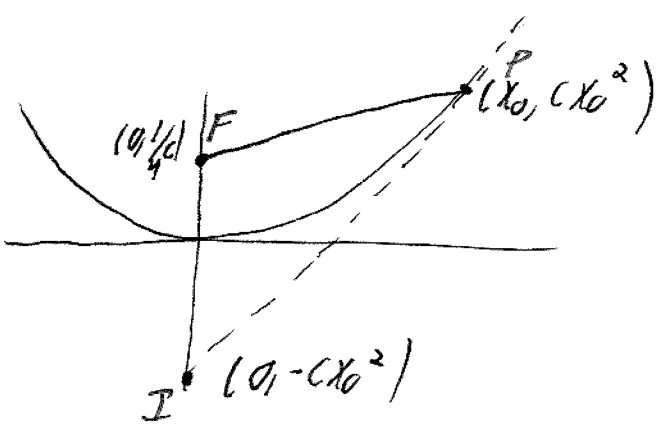
$$\boxed{y - cx_0^2 = 2cx_0 (x - x_0)}$$

rewriting as $\quad y = 2cx_0 x + cx_0^2 - 2cx_0^2$

$$y = 2cx_0 x - cx_0^2$$

$$\boxed{y \text{ intercept is } (0, -cx_0^2)}$$

c.



length $\overline{FI} = \frac{1}{4c} + cx_0^2$

length $\overline{FP} = \sqrt{x_0^2 + (cx_0^2 - \frac{1}{4c})^2} = \sqrt{x_0^2 + c^2x_0^4 - \frac{1}{2}x_0^2 + \frac{1}{16c^2}}$

$$= \sqrt{c^2x_0^4 + \frac{1}{2}x_0^2 + \frac{1}{16c^2}} = \sqrt{(cx_0^2 + \frac{1}{4c})^2}$$

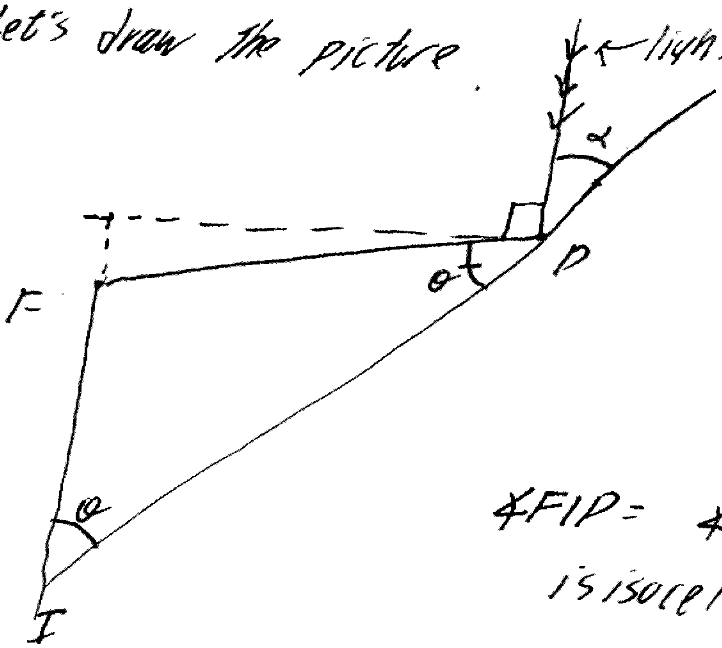
$$= (cx_0^2 + \frac{1}{4c})$$

c. length $\overline{PI} = \sqrt{x_0^2 + (2cx_0)^2} = \sqrt{4c^2x_0^2 + x_0^2}$

$$= \sqrt{(4c^2+1)x_0^2}$$

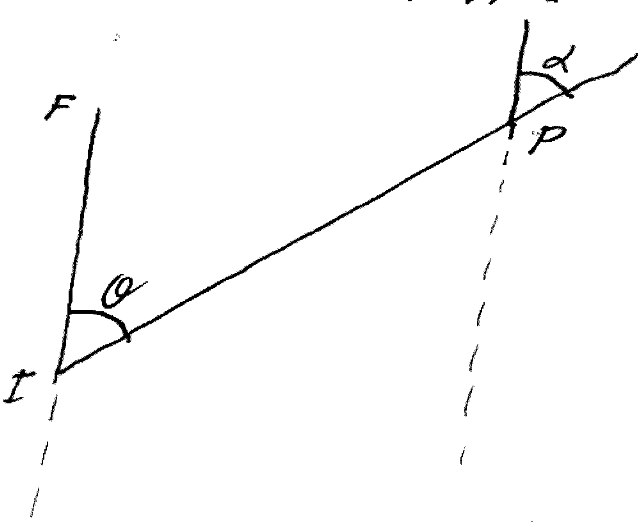
Thus $|\overline{FI}| = |\overline{FP}|$, the triangle is isosceles

d. Let's draw the picture.



$\measuredangle FIP = \measuredangle FPI$ since the triangle is isosceles

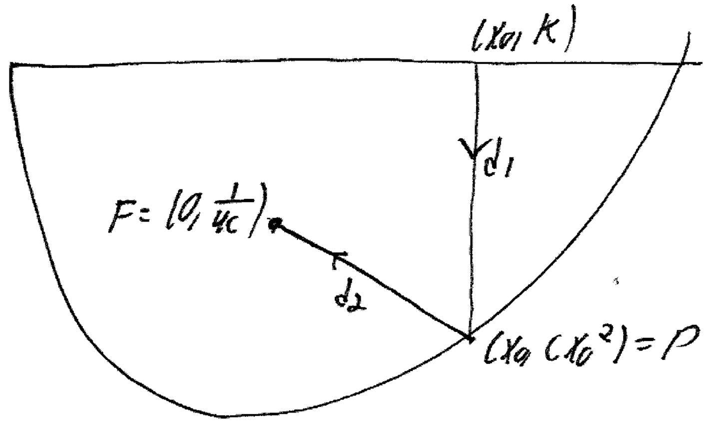
We must show $\measuredangle\alpha = \measuredangle\theta$ to obtain that the light bounces off P and to F. But this is clear if we instead look at $\theta = \measuredangle FIP$:



Since $\overline{FI}$ is parallel to the light,

$\measuredangle\alpha = \measuredangle\theta$.

e.



Choose $K > \frac{1}{4c}$, $K > cx_0^2$. We must show $d_1 + d_2$ does not depend on x_0.

$d_1 =$ ~~[illegible]~~ $K - cx_0^2$

$d_2 = |FP| = cx_0^2 + \frac{1}{4c}$ by part c.

Thus $d_1 + d_2 = K + \frac{1}{4c}$ does not depend on x_0.