p. 311 # 1, 2, 3, 5, 13, 14, 22

Review

L'Hospital's Rule Suppose $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ or

$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \pm\infty$ and $f(x), g(x)$ are differentiable

Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)} \quad \text{if it exists}$$

i.e. LHR works for "$\frac{0}{0}$" or "$\frac{\pm\infty}{\pm\infty}$"

Other types of limits where LHR can be used

1. "$0\cdot\infty$", i.e. $\lim_{x\to a} f(x)g(x)$ where $\lim_{x\to a} f(x) = 0$, $\lim_{x\to a} g(x) = \pm\infty$

   Write as $fg = \frac{f}{1/g}$ $\left(\frac{0}{0}\right)$ or $fg = \frac{g}{1/f}$ $\left(\frac{\infty}{\infty}\right)$

   and try LHR

2. "$\infty - \infty$", try to get $f(x) - g(x)$ as a quotient where LHR applies.

3. "0^0", "∞^0", "1^∞" indeterminate powers

   Try taking ln of each side, get

   $\lim_{x\to a} \ln y$ and then

   $$\lim_{x\to a} y = e^{\lim_{x\to a} \ln y}$$

Example

$$\lim_{x\to 0} (1-2x)^{1/x} \quad \text{type } "1^{\infty}"$$

$$y = (1-2x)^{1/x}$$

$$\ln y = \tfrac{1}{x}\ln(1-2x) = \frac{\ln(1-2x)}{x} \quad \text{type } \tfrac{0}{0}$$

$$\lim_{x\to 0} \ln y = \lim_{x\to 0} \frac{\ln(1-2x)}{x} = \lim_{x\to 0} \frac{\frac{-2}{1-2x}}{1} \text{ by LHR}$$

$$= -2$$

Thus as $x \to 0$, $\ln y \to -2$ so $y \Rightarrow e^{-2}$.

$$\boxed{\lim_{x\to 0} (1-2x)^{1/x} = e^{-2}}$$

Techniques of Integration

Review

FTOC $\int_a^b F'(x)\,dx = F(b) - F(a)$, thus we can integrate functions which we recognize as derivatives of something else.

* Every differentiation formula has a corresponding integral formula.

Example $y = \ln|\sin x|$ $y' = \frac{\cos x}{\sin x} = \cot x$ Thus

$$\boxed{\int \cot x\,dx = \ln|\sin x| + C}$$

Formulas so far are on page 306.

Suppose (as usual) we have an integral not on our list.

<u>Example</u> $\int \frac{x^2+2x}{x^3+3x^2+1}\,dx$

We make a substitution so it appears on this list!

$$u = x^3+3x^2+1 \quad du = 3x^2+6x$$

$$= \frac{1}{3}\int \frac{1}{u}\,du = \frac{1}{3}\ln|u| + C$$

$$= \frac{1}{3}\ln|x^3+3x^2+1| + C$$

Let $y = \frac{1}{3}\ln|x^3+3x^2+1|$ $\quad y' = \frac{1}{3}\cdot\frac{1}{x^3+3x^2+1}\cdot 3x^2+6x$

$$= \frac{x^2+2x}{x^3+3x^2+1}$$

** u-substitution is the <u>chain rule run in reverse</u> **

<u>Integration by Parts - Product Rule in Reverse</u>

<u>Recall</u> $(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$. Thus

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx$$

a.k.a. $\int u\,dv = uv - \int v\,du$

$$u = f(x) \quad v = g(x) \quad dv = g'(x)\,dx$$

Integration by Parts converts $\int u\,dv$ into $\int v\,du$,
hope is that $\int v\,du$ is simpler!

Example $\int xe^x\,dx$

$u = x \quad du = 1\,dx$

$dv = e^x dx \quad v = e^x dx$

$$\int u\,dv = uv - \int v\,du$$

$$= xe^x - \int e^x dx = xe^x - e^x + C$$

$$\boxed{\int xe^x dx = xe^x - e^x + C}$$

check by differentiating

Example $\int \ln x\,dx$

$u = \ln x \quad dv = dx$

$du = \frac{1}{x}dx \quad v = x$

$$\cancel{\int uv\,du} \quad \int u\,dv = uv - \int v\,du$$

$$= x\ln x - \int x \cdot \frac{1}{x}\,dx$$

$$= x\ln x - x$$

$$\boxed{\int \ln x\,dx = x\ln x - x}$$

Example

$$\int x^2 e^x dx$$

$u = x^2 \quad dv = e^x dx$

$du = 2x dx \quad v = e^x$

$$\int u\,dv = uv - \int v\,du$$

$$= x^2 e^x - \int 2xe^x$$

$$= x^2 e^x - 2 \cdot (xe^x - e^x) + C$$

$$\boxed{\int x^2 e^x dx = x^2 e^x - 2xe^x + 2e^x + C}$$

Example

$$\int e^x \cos x\,dx$$

$u = e^x \quad dv = \cos x\,dx$

$du = e^x dx \quad v = \sin x$

$$\int e^x \cos x\,dx = e^x \sin x - \int e^x \sin x\,dx$$

$u = e^x \quad dv = \sin x\,dx$

$du = e^x dx \quad v = -\cos x$

$$= e^x \sin x - \left(-e^x \cos x + \int e^x \cos x\,dx\right)$$

$$\int e^x \cos x\,dx = e^x \sin x + e^x \cos x - \int e^x \cos x\,dx$$

$$\int e^x \cos x\,dx = \frac{e^x \sin x + e^x \cos x}{2}$$