p 321 # 2, 6, 8, 11, 14, 19, 25, 26, 32

2/12/07

Trigonometric Integrals

Ex1 $\int \sin^5 x \cos x \, dx$ $\quad u = \sin x \quad du = \cos x \, dx$

$$= \boxed{\frac{1}{6}\sin^6 x + C}$$

Ex3 $\int \sin^5 x \, dx = \int (\sin^2 x)^2 \sin x \, dx$

$$= \int (1-\cos^2 x)^2 \sin x \, dx$$

$$= \int \cancel{\sin x \, dx} (\cos^4 x - 2\cos^2 x + 1) \sin x \, dx$$

$$= \boxed{-\frac{1}{5}\cos^5 x + \frac{2}{3}\cos^3 x - \cos x + C}$$

** To integrate $\int \sin^n x \cos^m x \, dx$ if n or m are odd, just "save" out one and turn the rest using $\cos^2 x + \sin^2 x = 1$

Ex $\int \cos^4 x \sin^7 x \, dx =$

$$\int \cos^4 x (\sin^2 x)^3 \sin x \, dx$$

$$= \int \cos^4 x (1-\cos^2 x)^3 \sin x \, dx$$

$$= \int -\cos^{12} x \sin x + 3\cos^{10} x \sin x - 3\cos^8 x \sin x + \cos^4 x \sin x \, dx$$

$$= \boxed{\frac{1}{13}\cos^{13} x - \frac{3}{11}\cos^{11} x + \frac{1}{3}\cos^9 x - \frac{1}{5}\cos^5 x + C}$$

<u>Problem</u> What about $\int \cos^n x \sin^m x \, dx$ n, m both even?

<u>Halfangle Formulas:</u>

$$\sin^2 x = \frac{1-\cos 2x}{2} \qquad \cos^2 x = \frac{1+\cos 2x}{2} \qquad \sin x \cos x = \tfrac{1}{2}\sin 2x$$

Use these to keep lowering the degree.

Ex
$$\int \cos^4 x \, dx = \int \left(\tfrac{1}{2}(1+\cos 2x)\right)^2 dx$$
$$= \frac{1}{4}\int \cos^2 2x + 2\cos 2x + 1 \, dx$$
$$= \frac{1}{4}\int \tfrac{1}{2} + \tfrac{1}{2}\cos(4x) + 2\cos 2x + 1 \, dx$$
$$= \boxed{\frac{1}{4}\left(\tfrac{1}{2}x + \tfrac{1}{8}\sin(4x) + \sin(2x) + x\right) + C}$$

Ex $\int \sin^2\theta \cos^2\theta \, d\theta$
$$= \int \left(\frac{1-\cos 2\theta}{2}\right)\left(\frac{1+\cos 2\theta}{2}\right) = \int \frac{1}{4}\left(1-\cos^2 2\theta\right)$$
$$= \int \frac{1}{4}\left(\sin^2 2\theta\right)$$
$$= \frac{1}{4}\cdot\frac{1}{2}\int 1-\cos(4\theta) \, d\theta$$
$$= \boxed{\frac{1}{8}\left(\theta - \tfrac{1}{4}\sin(4\theta)\right) + C}$$

$\int \tan^n\theta \sec^m\theta \, d\theta$

Recall $\sec^2 x = 1 + \tan^2 x$

$\frac{d}{dx}\sec x = \sec x \tan x$

$\frac{d}{dx}\tan x = \sec^2 x$

Example $\int \tan^3 x \sec^4 x \, dx$

$= \int \tan^3 x (1+\tan^2 x)\sec^2 x \, dx$

$= \int \tan^3 x \sec^2 x \, dx + \int \tan^5 x \sec^2 x \, dx$

$= \boxed{\frac{1}{4}\tan^4 x + \frac{1}{6}\tan^6 x + C}$

- save out $\sec^2 x$
- turn rest into tangents
- works for even powers of sec x

Example $\int \tan^3 x \sec^3 x \, dx$

- ODD POWER OF TAN plus some sec
- save out sec x tan x, rest into sec.

$= \int \tan^2 x \sec^2 x \cdot \sec x \tan x \, dx$

$= \int (\sec^2 x - 1)\sec^2 x \sec x \tan x \, dx$

$= \int \sec^4 x \cdot \sec x \tan x \, dx - \int \sec^2 x \cdot \sec x \tan x \, dx$

$= \boxed{\frac{1}{5}\sec^5 x - \frac{1}{3}\sec^3 x + C}$

What about other cases?

$\int \tan x\,dx = \ln|\sec x| + C \qquad \int \sec x\,dx = \ln|\sec x + \tan x| + C$

Ex $\int \tan^3 x\,dx = \int \tan x(\sec^2 x - 1)$

$= \int \tan x \sec^2 x\,dx - \int \tan x\,dx$

$= \boxed{\frac{1}{2}\tan^2 x - \ln|\sec x| + C}$

Ex

$\int \sec^3 x\,dx$

$u = \sec x \qquad dv = \sec^2 x\,dx$

$du = \cancel{\tan}\ \sec x \tan x\,dx \qquad v = \tan x$

$= \sec x \tan x - \int \sec x \overset{\sec^2 x - 1}{\tan^2 x}$

$= \sec x \tan x - \int \sec^3 x\,dx + \int \sec x\,dx$

$2\int \sec^3 x\,dx = \sec x \tan x + \ln|\sec x + \tan x| + C'$

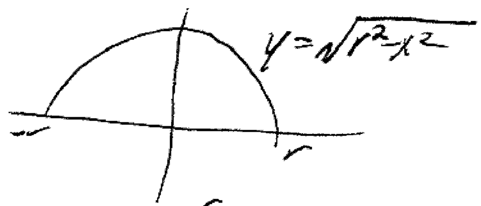
$\boxed{\int \sec^3 x\,dx = \frac{1}{2}(\sec x \tan x + \ln|\sec x + \tan x|) + C}$

** Of course similar tricks work for

$\csc^n x \cot^n x$

TRIG SUBSTITUTION

Problem Find area of a circle radius r



$$\text{Area} = \int_{-r}^{r} \sqrt{r^2 - x^2}\, dx$$

$$x = r\sin\theta \quad dx = r\cos\theta\, d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \sqrt{r^2(1-\sin^2\theta)} \cdot r\cos\theta\, d\theta$$

$$= \int_{-\pi/2}^{\pi/2} r^2 \cos^2\theta\, d\theta$$

$$= \int_{-\pi/2}^{\pi/2} r^2\left(\tfrac{1}{2} + \tfrac{1}{2}\cos 2\theta\right) d\theta$$

$$= \frac{r^2}{2}\theta + \frac{1}{4}\sin 2\theta \Big|_{-\pi/2}^{\pi/2}$$

$$= \frac{r^2}{2}\cdot\left(\pi/2 - -\pi/2\right) =$$

$$\pi r^2/2$$

$$\text{Area } \bigcirc = \pi r^2$$