p. 321 # 42, 52, 51, 56, 57

2/13/07

Recall $\cos^2\theta = 1 - \sin^2\theta$ $\qquad$ $\sec^2\theta = 1 + \tan^2\theta$

$\sec^2\theta - 1 = \tan^2\theta$

The identities plus "reverse u-substitution" let us do integrals with $\sqrt{\pm a^2 \pm x^2}$ in them.

Example

$\int \sqrt{r^2 - x^2}\, dx$ $\qquad$ set $x = r\sin\theta$ $\quad dx = r\cos\theta\, d\theta$

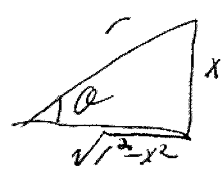
$= \int \sqrt{r^2 - r^2\sin^2\theta} \cdot r\cos\theta\, d\theta$

$= \int r\sqrt{1-\sin^2\theta}\, r\cos\theta\, d\theta$

$= \int r^2\cos^2\theta\, d\theta = r^2 \int \frac{1}{2}(1+\cos 2\theta)\, d\theta$

$= \frac{r^2}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right) + C$

Now $\theta = \sin^{-1}\left(\frac{x}{r}\right)$,



$\sin 2\theta = 2\sin\theta\cos\theta$

$= \frac{r^2}{2}\left(\sin^{-1}\left(\frac{x}{r}\right) + \frac{x}{r} \cdot \frac{\sqrt{r^2-x^2}}{r}\right)$

$= \frac{r^2}{2}\sin^{-1}\left(\frac{x}{r}\right) + \frac{x\sqrt{r^2-x^2}}{2} + C$

Summary

$\sqrt{a^2 - x^2}$ $\qquad x = a\sin\theta$

$\sqrt{a^2 + x^2}$ $\qquad x = a\tan\theta$

$\sqrt{x^2 - a^2}$ $\qquad x = a\sec\theta$

Example

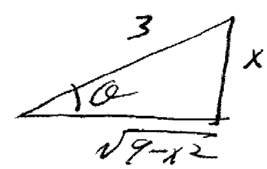
$$\int \frac{\sqrt{9-x^2}}{x^2}\,dx \qquad x = 3\sin\theta \quad dx = 3\cos\theta\, d\theta$$

$$= \int \frac{\sqrt{9-9\sin^2\theta}}{9\sin^2\theta}\cdot 3\cos\theta\, d\theta = \int \frac{9\cos^2\theta}{9\sin^2\theta} = \int \cot^2\theta\, d\theta$$

$$= \int \csc^2\theta - 1\, d\theta = -\cot\theta - \theta + C$$

Now $\theta = \sin^{-1}\left(\frac{x}{3}\right)$

$\cot\theta = \frac{\sqrt{9-x^2}}{x}$



$$= \boxed{-\frac{\sqrt{9-x^2}}{x} - \sin^{-1}\left(\frac{x}{3}\right) + C}$$

Example
#45

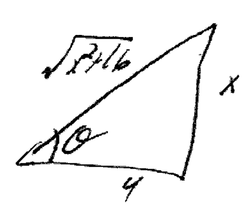
$$\int \frac{dx}{\sqrt{x^2+16}} \qquad x = 4\tan\theta \quad dx = 4\sec^2\theta\, d\theta$$

$$= \int \frac{4\sec^2\theta}{\sqrt{16(1+\tan^2\theta)}}\, d\theta = \int \frac{4\sec^2\theta}{4\sec\theta}\, d\theta$$

$$= \int \sec\theta\, d\theta = \ln|\sec\theta + \tan\theta| + C$$

Now $\theta = \tan^{-1}\left(\frac{x}{4}\right)$

$\sec\theta = \frac{\sqrt{x^2+16}}{4}$ $\quad\tan\theta = \frac{x}{4}$



$$= \ln\left|\frac{\sqrt{x^2+16}}{4} + \frac{x}{4}\right| + C$$

$$= \ln\left|\sqrt{x^2+16} + x\right| - \ln 4 + C$$

$$\boxed{= \ln\left|\sqrt{x^2+16} + x\right| + C}$$

Example

#52 $\displaystyle\int \frac{dx}{x^2\sqrt{16x^2-9}} = \int \frac{dx}{4x^2\sqrt{x^2-9/16}}$

$x = \frac{3}{4}\sec\theta \quad dx = \frac{3}{4}\sec\theta\tan\theta\, d\theta$

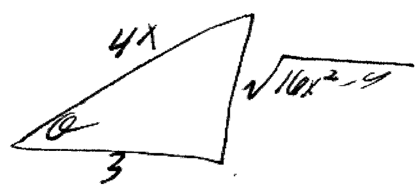
$$= \int \frac{3/4 \sec\theta\tan\theta\, d\theta}{\frac{9}{4}\sec^2\theta\sqrt{\frac{9}{16}\sec^2\theta - 9/16}}$$

$$= \int \frac{3/4 \sec\theta\tan\theta\, d\theta}{9/4\sec^2\theta \cdot \frac{3}{4}\cdot\tan\theta}$$

$$= \int \frac{4}{9}\cdot\frac{1}{\sec\theta}\, d\theta = \frac{4}{9}\sin\theta + C$$

$\theta = \sec^{-1}\left(\frac{4x}{3}\right)$

$\sin\theta = \dfrac{\sqrt{16x^2-9}}{4x}$

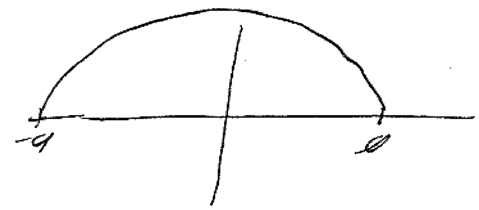


$$= \frac{4}{9}\,\frac{\sqrt{16x^2-9}}{4x} + C = \boxed{\frac{\sqrt{16x^2-9}}{9x} + C}$$

Example

Find area of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Solution $y=\frac{b}{a}\sqrt{a^2-x^2}$



Area $= 2\int_{-a}^{a}\frac{b}{a}\sqrt{a^2-x^2}\,dx = \frac{b}{a}2\cdot\left(\frac{a^2}{2}\right)\left(\sin^{-1}\left(\frac{x}{a}\right)+\frac{x\sqrt{a^2-x^2}}{2}\right)\Big|_{-a}^{+a}$

$= ab\cdot\left(\sin^{-1}(1)+0-\sin^{-1}(-1)+0\right)$

$= ab(\pi) = \pi ab$

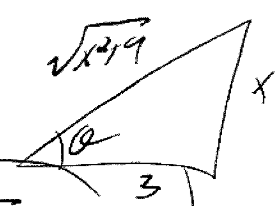
Example $\int \frac{x}{\sqrt{x^2+9}}\,dx$

METHOD 1

$x=3\tan\theta \quad dx=3\sec^2\theta\,d\theta$

$=\int\frac{3\tan\theta\cdot 3\sec^2\theta\,d\theta}{3\sec\theta}=\int 3\sec\theta\tan\theta = 3\sec\theta+C$

$\theta=\tan^{-1}\left(\frac{x}{3}\right) \quad \sec\theta=\frac{\sqrt{x^2+9}}{3}$



$=\sqrt{x^2+9}+C$

METHOD 2

$u=x^2+9$

$du=2x\,dx$

$=\int\frac{1}{2}u^{-1/2}\,du = u^{1/2}+C = \sqrt{x^2+9}+C$