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20, 30, 40

2/14/07 Partial Fractions

Integrals We will need

- $\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

- $\int \frac{1}{u}\,du = \ln|u| + C$

Example $\int \frac{6x+5}{x^2+7} = \int \frac{6x}{x^2+7}\,dx + \int \frac{5}{x^2+7}\,dx$

$= 3\ln|x^2+7| + \frac{5}{\sqrt{7}}\tan^{-1}\left(\frac{x}{\sqrt{7}}\right) + C$

Completing the square

$x^2+bx+c = \left(x+\frac{b}{2}\right)^2 + c - \frac{b^2}{4}$

Ex $x^2+7x-5 = \left(x+\frac{7}{2}\right)^2 - \frac{69}{4}$

$3x^2+2x+1 = 3\left(x^2+\frac{2}{3}x+\frac{1}{3}\right)$

$= 3\left(\left(x+\frac{1}{3}\right)^2 + \frac{2}{9}\right)$

Integrating $\frac{p(x)}{q(x)}$

1. If $\deg p(x) \geq \deg q(x)$ use long division to write as a polynomial + fraction w/ degree on top < degree bottom

2. Factor bottom into irreducible linear and quadratics

3. Partial Fractions

<u>Example</u> $\displaystyle\int \frac{x^2+x-2}{x^3+9x}\,dx$

$$\frac{x^2+x-2}{x(x^2+9)} = \frac{A}{x} + \frac{Bx+C}{x^2+9}$$ solve for B, C

$$x^2+x-2 = A(x^2+9) + (Bx+C)x$$ can plug in x values or multiply out

$x=0 \quad -2 = 9A \qquad A = -2/9$

x term $\quad x = Cx \qquad C = 1$

x^2 term $\quad x^2 = Ax^2 + Bx^2 \qquad A+B=1 \quad B = 11/9$

$$= \int \frac{-2/9}{x} + \frac{11/9\,x + 1}{x^2+9}\,dx$$

$$= -2/9 \ln|x| + \int \frac{11}{9}\cdot\frac{x}{x^2+9} + \int \frac{1}{x^2+9}$$

$$= \boxed{-2/9 \ln|x| + \frac{11}{18}\ln|x^2+9| + \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) + C}$$

Repeat Factors in Denom

If repeats then partial fractions up to highest degree.

Example $\int \frac{x}{x^3-x^2-x+1}\,dx = \int \frac{x}{(x-1)^2(x+1)}\,dx$

$$\frac{x}{(x-1)^2(x+1)} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{x+1}$$

$$x = A(x-1)(x+1) + B(x+1) + C(x-1)^2$$

$x=1 \rightarrow \quad B = 1/2$

$x=-1 \rightarrow \quad -1 = 4C \quad C = -1/4$

x^2 coef $\quad 0 = A + C \quad A = 1/4$

$$= \int \frac{1/4}{x-1} + \frac{1/2}{(x-1)^2} \quad \frac{-1/4}{x+1}\,dx$$

$$= \boxed{1/4 \ln|x-1| - \frac{1}{2}(x-1)^{-1} - \frac{1}{4}\ln|x+1| + C}$$

Example #5

$$\int \frac{4x^2-3x+2}{4x^2-4x+3}\,dx = \int 1 + \frac{x-1}{4x^2-4x+3}\,dx \qquad b^2-4ac<0$$

$$4x^2-4x+3 = (2x-1)^2+2$$

$$\text{so} \quad u=2x-1 \qquad du = 2\,dx$$

$$x=(u+1)/2 = \tfrac{1}{2}u+\tfrac{1}{2}$$

$$= x + \tfrac{1}{2}\int \cancel{u-1/2}$$

$$x + \int \frac{\frac{1}{2}u - 1/2}{u^2+2}\cdot\tfrac{1}{2}\,du$$

$$= x + \tfrac{1}{4}\int \frac{u-1}{u^2+2}\,du$$

$$= x + \tfrac{1}{4}\int \frac{u}{u^2+2} - \tfrac{1}{4}\int \frac{1}{u^2+2}$$

$$= x + \tfrac{1}{8}\ln|u^2+2| - \frac{1}{4\sqrt{2}}\tan^{-1}\left(\frac{u}{\sqrt{2}}\right) + C$$

$$= \boxed{x + \tfrac{1}{8}\ln|4x^2-4x+3| - \frac{1}{4\sqrt{2}}\tan^{-1}\left(\frac{2x-1}{\sqrt{2}}\right) + C}$$

General $\dfrac{Ax+B}{ax^2+bx+c}$ with $b^2-4ac<0$

1. Complete square to get

$$\int \frac{Cu+D}{u^2+e^2} = C\int \frac{u}{u^2+e^2} + D\int \frac{1}{u^2+e^2}$$

$$\uparrow \qquad\qquad \uparrow$$

$$\ln \qquad\qquad \tan^{-1}$$

Example Write out partial fraction decomp:

$$\frac{x^3+2x+1}{x(x+2)^2(x^2+x+3)(x^2+1)^3}$$

$$= \frac{A}{x} + \frac{B}{(x+2)} + \frac{C}{(x+2)^2} + \frac{Dx+E}{x^2+x+3} + \frac{Fx+G}{(x^2+1)} + \frac{Hx+I}{(x^2+1)^2} + \frac{Jx+K}{(x^2+1)^3}$$