p. 335 # 6, 8, 23a, 15
p. 354 # 6, 10, 13, 19, 26, 27, 29

2/22/07

Review

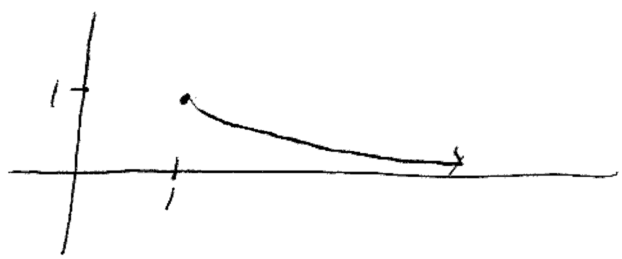
Definite integral $\int_a^b f(x)\,dx$ was defined as a limit of Riemann sums, if the limit exists.

Thm If $f(x)$ is continuous on $[a,b]$, or even if finitely many jump discontinuities, then
$$\int_a^b f(x)\,dx \text{ exists}$$

Goal See if we can find integral over an infinite length or perhaps with an infinite discontinuity.

Case 1 Infinite integrals

Example Find area under $y = 1/x^2$ and above $[1,\infty)$



Step 1 Let $A(t)$ be area under $y = 1/x^2$ above $[1, t]$

$$A(t) = \int_1^t \frac{1}{x^2}\,dx = -\frac{1}{x}\Big]_1^t = \frac{-1}{t} + 1 = 1 - \frac{1}{t}$$

As $t \to \infty$ this $\to 1$

Thus we say

$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^2}\,dx = 1$$

total area under curve is 1.

Def. Suppose $\int_a^t f(x)\,dx$ exists for all $t \geq a$. Then

1. $$\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx \quad \text{if it exists}$$

2. Similarly define $$\int_{-\infty}^b = \lim_{t\to-\infty} \int_t^b f(x)\,dx \quad \text{if it exists}$$

called improper integrals
If limit exists, call integrals convergent.

3. Finally if $\int_a^\infty f(x)\,dx$ & $\int_{-\infty}^a f(x)\,dx$ exist then

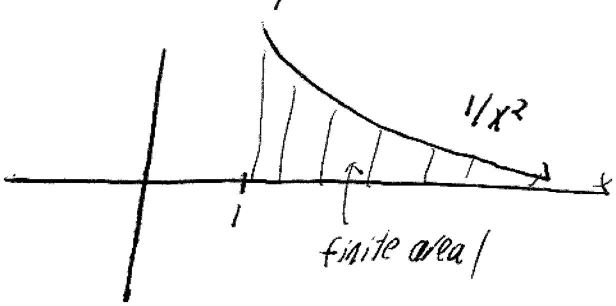
Define $$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx.$$

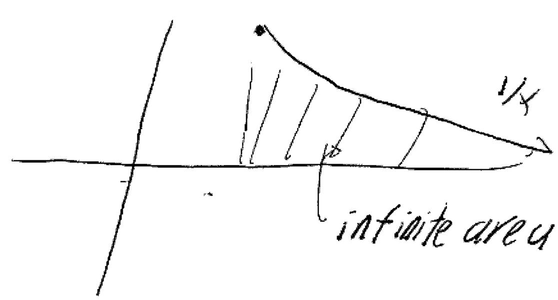
In #3 any a works

Example

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x}\,dx = \lim_{t\to\infty} \ln|x|\Big|_1^t = \lim_{t\to\infty} \ln t = \infty$$

$\int_1^\infty \frac{1}{x^2}\,dx$ converges $\qquad \int_1^\infty \frac{1}{x}\,dx$ diverges





Question $p > 0$, when is $\int_1^\infty \frac{1}{x^p}\,dx$ convergent.

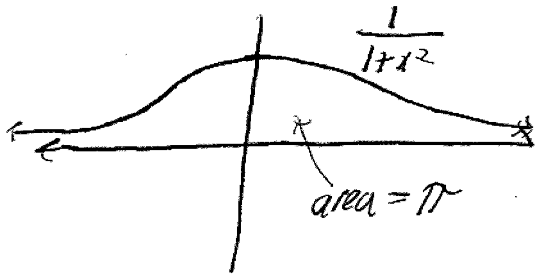
Ex $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx + \int_{0}^{\infty} \frac{1}{1+x^2}\,dx$

$$= \lim_{t\to-\infty} \int_{t}^{0} \frac{1}{1+x^2}\,dx + \lim_{t\to\infty} \int_{0}^{t} \frac{1}{1+x^2}\,dx$$

$$= \lim_{t\to-\infty} \tan^{-1}x \Big|_{t}^{0} + \lim_{t\to\infty} \tan^{-1}x \Big|_{0}^{t}$$

$$= \lim_{t\to-\infty} \tan^{-1}0 - \tan^{-1}t + \lim_{t\to\infty} \tan^{-1}t - \tan^{-1}0$$

$$= --\pi/2 + \pi/2 = \pi.$$



Ex $\displaystyle\int_{0}^{\infty} s e^{-5s}\,ds = \lim_{t\to\infty} \int_{0}^{t} s e^{-5s}\,ds$

$u = s \quad dv = e^{-5s}$

$du = ds \quad v = -\frac{1}{5} e^{-5s}$

$$\int s e^{-5s}\,ds = -\frac{1}{5} s e^{-5s} + \frac{1}{5}\int e^{-5s}\,ds$$

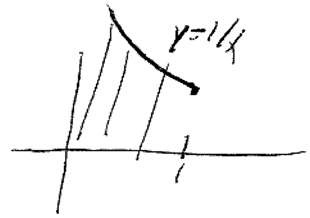
$$= -\frac{1}{5} s e^{-5s} - \frac{1}{25} e^{-5s}$$

$$\lim_{t\to\infty}\left(\frac{-t}{e^{5t}} - \frac{1}{25e^{5t}}\right) - \left(0 - 1/25\right)$$

$$= 1/25$$

Type 2 Suppose f is almost continuous on closed interval except a vertical asymptote at end

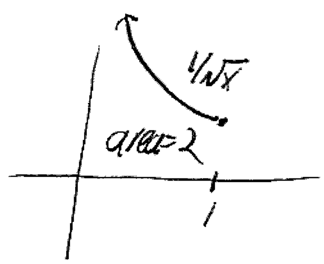
Ex 1 $\int_0^1 \frac{1}{x}dx$



$$\int_t^1 \frac{1}{x}dx = \ln|x|\Big|_t^1 = -\ln t$$

Let $t \to 0$, $-\ln t \to \infty$. Thus infinite area.

Ex 2 $\int_0^1 \frac{1}{\sqrt{x}}dx = \lim_{t\to 0} \int_t^1 x^{-1/2}dx = \lim_{t\to 0} 2\sqrt{x}\Big|_t^1 = \lim_{t\to 0} 2-2\sqrt{t} = 2$



Def 1. If f is continuous on $[a,b)$ and discontinuous at b then

$$\int_a^b f(x)\,dx = \lim_{t\to b^-} \int_a^t f(x)\,dx \quad \text{if it exists}$$

2. Same on $(a,b]$ $\quad \int_a^b f(x)\,dx = \lim_{t\to a^+} \int_t^b f(x)\,dx$ if it exists

3. If ~~...~~ discontinuous at c, $a<c<b$ then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \quad \text{if exists}$$

Ex $\int_{2}^{5} \frac{1}{x-4}\, dx$

Ex $\int_{0}^{1} \frac{3}{x^5}\, dx$

Ex $\int_{0}^{1} \frac{dx}{\sqrt{1-x^2}}\, dx$