1. $\int \frac{x^4+1}{x(x^2+x+1)^2}\,dx$

$$\frac{x^4+1}{x(x^2+x+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+x+1} + \frac{Dx+E}{(x^2+x+1)^2}$$

$$x^4+1 = A(x^2+x+1)^2 + (Bx^2+Cx)(x^2+x+1) + Dx^2+Ex$$

const. term $\quad 1 = A$

x term $\quad 0 = 2A + C + E$

x^2 term $\quad 0 = 3A + B + C + D$ $\Rightarrow$ $A=1,\ B=0,\ C=-2,\ E=0,\ D=-1$

x^3 term $\quad 0 = 2A + B + C$

x^4 term $\quad 1 = A + B$

$$= \int \frac{1}{x} - \frac{2}{x^2+x+1} - \frac{x}{(x^2+x+1)^2}\,dx$$

$x^2+x+1 = (x+1/2)^2 + 3/4$

$u = x+1/2 \quad du = dx$

$x = u - 1/2$

$$= \ln|x| - 2\int \frac{1}{u^2+3/4}\,du - \int \frac{u-1/2}{(u^2+3/4)^2}\,du$$

$$= \ln|x| - 2\cdot\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2u}{\sqrt{3}}\right) - \int \frac{u}{(u^2+3/4)^2} + \frac{1}{2}\int \frac{1}{(u^2+3/4)^2}\,du$$

$$= \ln|x| - \frac{4}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + \frac{1}{2}(u^2+3/4)^{-1} + \frac{1}{2}\int \frac{1}{(u^2+3/4)^2}\,du$$

$$= \ln|x| - \frac{4}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + \frac{1}{2(x^2+x+1)} + \frac{1}{2}\int \frac{1}{(u^2+3/4)^2}\,du$$

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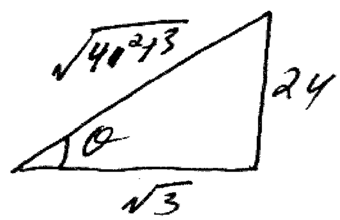
1. (cont)

$$\int \frac{1}{(u^2+3/4)^2}\,du \qquad u = \frac{\sqrt{3}}{2}\tan\theta \quad du = \frac{\sqrt{3}}{2}\sec^2\theta$$

$$u^2+3/4 = 3/4\,(1+\tan^2\theta) = 3/4\sec^2\theta$$

$$= \int \frac{\frac{\sqrt{3}}{2}\sec^2\theta}{9/16\sec^4\theta}\,d\theta = \frac{8\sqrt{3}}{9}\int \cos^2\theta\,d\theta = \frac{8\sqrt{3}}{9}\cdot\frac{1}{2}\cdot(\theta+\cos\theta\sin\theta)$$

$$\theta = \tan^{-1}\left(\frac{2u}{\sqrt{3}}\right)$$



$$\cos\theta = \frac{\sqrt{3}}{\sqrt{4u^2+3}} \qquad \sin\theta = \frac{2u}{\sqrt{4u^2+3}}$$

$$= \frac{4\sqrt{3}}{9}\left(\tan^{-1}\left(\frac{2u}{\sqrt{3}}\right) + \frac{2\sqrt{3}\,u}{4u^2+3}\right)$$

$$= \frac{4\sqrt{3}}{9}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + \frac{2\sqrt{3}\cdot(x+1/2)}{4x^2+4x+1}$$

<u>Total answer</u>

$$\ln|x| - \frac{4}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + \frac{1}{2(x^2+x+1)} + \frac{2\sqrt{3}}{9}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + \frac{\sqrt{3}x+\sqrt{3}}{4(x^2+x+1)}$$

$$= \boxed{\ln|x| - \frac{10\sqrt{3}}{9}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + \frac{\sqrt{3}x+\sqrt{3}+2}{4(x^2+x+1)} + C}$$

2. $\displaystyle\int_{1/3}^{3} \frac{\sqrt{x}}{x^2+x}\,dx \qquad u=\sqrt{x} \qquad du=\frac{1}{2\sqrt{x}}\,dx$

$$dx = 2\sqrt{x}\,du = 2u\,dy$$

$$= \int \frac{u\cdot 2u\,dy}{u^4+u^2} = \int \frac{2}{u^2+1}\,dy$$

$$= 2\tan^{-1}(u)$$

$$= 2\tan^{-1}(\sqrt{x})\Big|_{1/3}^{3}$$

$$= 2\left(\tan^{-1}(\sqrt{3}) - \tan^{-1}(1/3)\right)$$