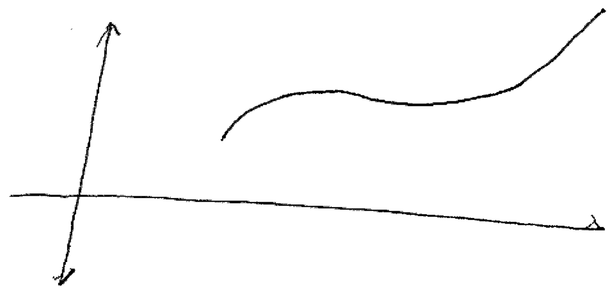
p. 385
#3, 7, ~~10~~ 13, 15

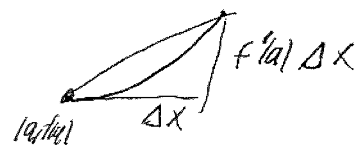
#3,
7: Hint algebra → perf squares under $\sqrt{\ }$
13 Hint $u=e^x$ then use tables

2/27/07 **7.4 Arc Length**

Consider graph $y=f(x)$



How long is this curve?



length curve $\approx$ length of line segment $= \sqrt{\Delta x^2 + (f'(a)\Delta x)^2}$

$= \sqrt{1+f'(a)}\ \Delta x$

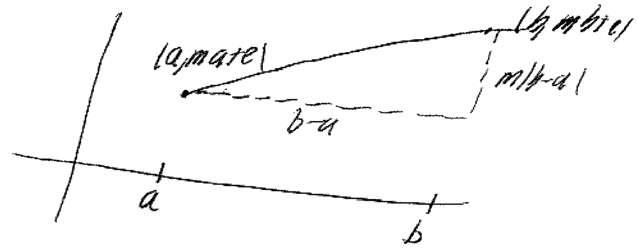
Add up length $\approx \sum_{i=1}^{n} \sqrt{1+f'(x_i)}\ \Delta x_i$

Theorem Suppose $f'(x)$ is continuous on $[a,b]$. Then the length of the curve $y=f(x)$ $a \le x \le b$ is

$$L = \int_a^b \sqrt{1+(f'(x))^2}\,dx$$

$$= \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

Example 1 $y = mx + e$



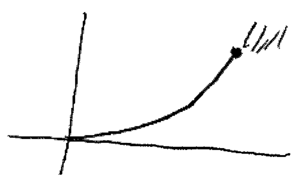
$\sqrt{m^2(b-a)^2+(b-a)^2}$

$$L = \int_a^b \sqrt{1+m^2}\,dx = \sqrt{1+m^2}\; x \Big|_a^b$$

$$= (b-a)\cdot\sqrt{1+m^2}$$

✓ matches Pythag Thm

Example 2 Parabola $y=x^2$ from $(0,0)$ to $(1,1)$



$$L = \int_0^1 \sqrt{1+(2x)^2}\,dx$$

$$= \int_0^1 \sqrt{1+4x^2}\,dx$$

$x = \frac{1}{2}\tan\theta \quad dx = \frac{1}{2}\sec^2\theta\,d\theta$

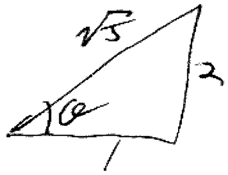
$x=0 \quad \theta=0$

$x=1 \quad \theta=\tan^{-1}2$

$$\int\sqrt{1+4x^2}\,dx = \int\sqrt{1+\tan^2\theta}\cdot\frac{1}{2}\sec^2\theta\,d\theta$$

$$= \frac{1}{2}\int\sec^3\theta\,d\theta$$

$$= \frac{1}{2}\left(\frac{1}{2}\right)\left(\sec\theta\tan\theta + \ln|\sec\theta+\tan\theta|\,\right)\Big|_0^{\tan^{-1}2}$$



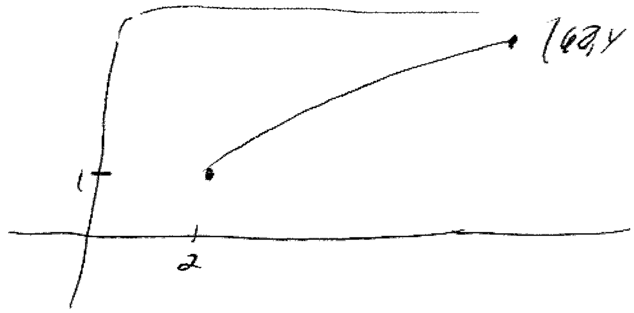
$$= \frac{1}{4}\left(2\sqrt{5} + \ln|2+\sqrt{5}|\right) - (0+\ln 1)$$

$$= \frac{\sqrt{5}}{2} + \frac{\ln|2+\sqrt{5}|}{4}$$

$\sec\theta = \sqrt{5}$

$\tan\theta = 2$

EX $x = y + y^3 \quad 1 \le y \le 4$



Thm Curve is $x = g(y)$ $c \le y \le d$ then length is

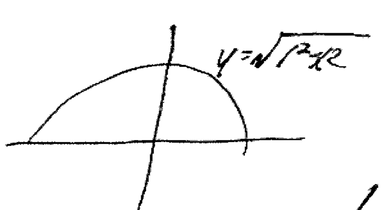
$$L = \int_c^d \sqrt{1 + g'(y)^2}\, dy = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

A: $$\int_1^4 \sqrt{1 + (1+3y^2)^2}\, dy = \int_1^4 \sqrt{2 + 6y^2 + 9y^4}\, dy$$

$= ???$ MAPLE GIVES ANSWER AS CLOSE AS WE WANT!

Rmk Most times the integral $\int \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$ will be not easy!

Example Find circumference of a circle, radius r.



$$y' = \frac{-2x}{2\sqrt{r^2 - x^2}} = \frac{-x}{\sqrt{r^2 - x^2}}$$

$$\text{Circ} = 2 \cdot \int_{-r}^{r} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

$$= 2 \int_{-r}^{r} \sqrt{1 + \frac{x^2}{r^2 - x^2}} = 2 \int_{-r}^{r} \sqrt{\frac{r^2}{r^2 - x^2}}\, dx$$

Let $x = r\sin\theta$ $\quad x=-r \quad \theta=-\pi/2$

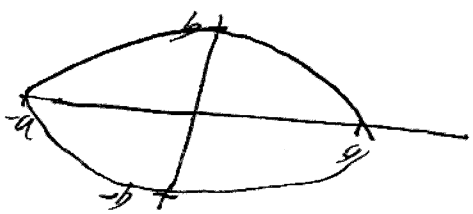
$dx = r\cos\theta\, d\theta$ $\quad x=r \quad \theta=\pi/2$

$$= 2\int_{-\pi/2}^{\pi/2}\sqrt{\frac{r^2}{r^2(1-\sin^2(\theta))}}\cdot r\cos\theta\, d\theta$$

$$= 2\int_{-\pi/2}^{\pi/2}\frac{1}{\cos\theta}\cdot r\cos\theta\, d\theta = 2\int_{-\pi/2}^{\pi/2} r\,d\theta = 2r\theta\Big|_{-\pi/2}^{\pi/2}$$

$$= 2\pi r = \pi D$$

<u>Example</u> Perimeter of ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$



$\frac{y^2}{b^2} = 1-\frac{x^2}{a^2}$

$y^2 = b^2-\frac{b^2}{a^2}x^2 \qquad y=\sqrt{b^2-\frac{b^2}{a^2}x^2}$

$y = b\sqrt{1-\frac{x^2}{a^2}}$

Per = 2

Per =

$y' = b\cdot\frac{1}{2\sqrt{1-\frac{x^2}{a^2}}}\cdot\frac{-2}{a^2}x$

$y' = \frac{-bx}{\sqrt{1-x^2/a^2}}$

$$P = 2\int_{-a}^{a}\sqrt{1+\frac{b^2x^2}{1-x^2/a^2}} = 2\int_{-a}^{a}\sqrt{\frac{1+(b^2-\frac{1}{a^2})x^2}{1-\frac{x^2}{a^2}}}$$

$$= 2\int_{-a}^{a}\sqrt{\frac{a^2+(b^2a^2-1)x^2}{a^2-x^2}}$$

<u>elliptic integral</u>

Examples

#9 $y = \ln|\sec x|$ $0 \le x \le \pi/4$

Find length

$$y' = \frac{1}{\sec x} \sec x \tan x$$

$$\int_0^{\pi/4} \sqrt{1+\tan^2 x}\, dx$$

$$= \int_0^{\pi/4} \sec x\, dx = \cancel{\tan x \Big|_0^{\pi/4} = 1}$$

$$= \ln|\sec x + \tan x| \Big|_0^{\pi/4} = \ln(\sqrt{2}+1) - \ln(1+0)$$

$$= \ln|\sqrt{2}+1|$$

#15

SET up integral for length of $y = 2^x$ $0 \le x \le 3$

$$y' = 2^x \ln 2$$

$$\int_0^3 \sqrt{1+(2^x \ln 2)^2}\, dx$$

$$= \int_0^3 \sqrt{1+2^{2x}(\ln 2)^2}\, dx$$