HW p. 446 #2, 4, 5, 7, 13, 14, 21, 25, 26, 33, 36, 37, 40, 41, 48

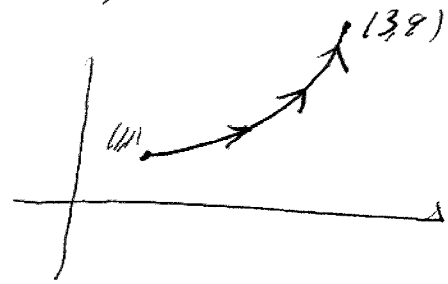
3/12/07 **Calculus of Parametric Curves**

Review

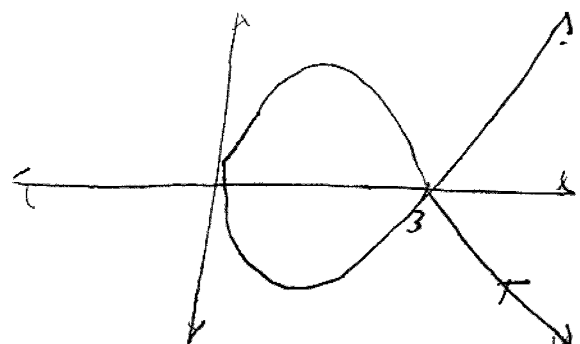
Given: $\vec{r}(t) = (x(t), y(t))$ a parametrized curve.

$$\vec{r}: \mathbb{R} \longrightarrow \mathbb{R}^2$$

Example: 1 $\vec{r}(t) = (t, t^2)$ $1 \le t \le 3$



Example 2 $\vec{r}(t) = (t^2, t^3 - 3t)$



Notice $r(\sqrt{3}) = r(-\sqrt{3}) = (3,0)$ crosses itself.

Definition Let $\vec{r}(t) = (x(t), y(t))$ be a parameterized curve. The

velocity vector is $\vec{v}(t) = (x'(t), y'(t))$

$$= \left(\frac{dx}{dt}, \frac{dy}{dt}\right).$$

This points in the direction the particle is traveling

$(\vec{v}(t)$ AKA $\vec{r}'(t))$

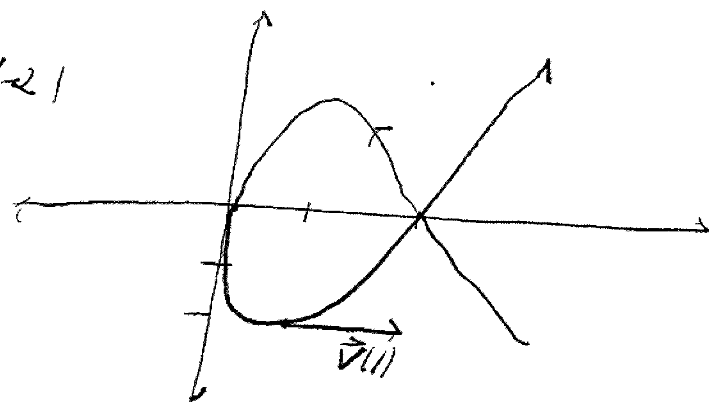
<u>Example</u> $\vec{r}(t) = (t^2, t^3 - 3t)$

1. Find $\vec{v}(t)$.
2. Find velocity vector at time $t=1$
3. Find eq of tangent line at $t=1$
4. Find tangent line at point $(3,0)$.

<u>Solution</u>

1. $\vec{v}(t) = (2t, 3t^2 - 3)$

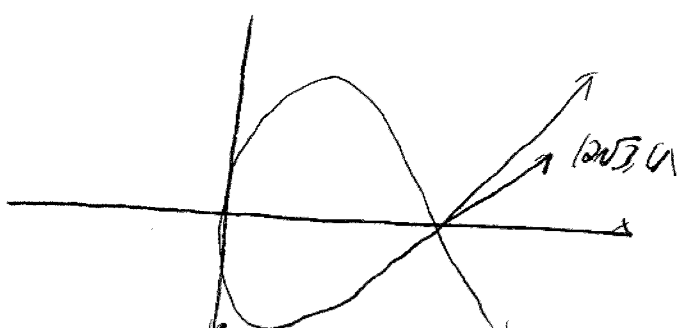
2. $\vec{v}(1) = (2, 0)$ $\quad r(1) = (1, -2)$



3 Horizontal so $y = -2$

4. $(3,0) = (t^2, t^3 - 3t) \rightarrow t = \pm\sqrt{3}$

$\vec{v}(\sqrt{3}) = (2\sqrt{3}, 6)$ $\quad \vec{v}(-\sqrt{3}) = (-2\sqrt{3}, 6)$



$$\text{slope} = \frac{\text{RISE}}{\text{RUN}} = \frac{y'(t)}{x'(t)} = \frac{6}{2\sqrt{3}} = \sqrt{3}$$

Two answers: $y = \sqrt{3}(x-3)$

$y = \sqrt{3}(x-3)$ particle passes through twice

Thm Given $\vec{r}(t) = (x(t), y(t))$, the slope of the tangent line to the curve at $t = t_0$ is

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{y'(t_0)}{x'(t_0)}.$$

Example

$$x(t) = e^{t^2+1} \qquad y(t) = t^3 + \cos t \qquad r(t) = (e^{t^2+1}, t^3 + \cos t)$$

Find eq of tangent line at $t = 1$

$$x'(t) = 2te^{t^2+1} \qquad y'(t) = 3t^2 - \sin t$$

point: $r(1) = (e^2, 2)$ $\quad$ slope $= \dfrac{y'(1)}{x'(1)} = \dfrac{3 - \sin 1}{2e^2}$

$$\boxed{y - 2 = \frac{3 - \sin 1}{2e^2}(x - e^2)}$$

Speed

Def The **speed** is the length of the velocity vector:

$$\text{speed} = |\vec{v}(t)| = |r'(t)| = \sqrt{x'(t)^2 + y'(t)^2}$$

$$= \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

Arc Length

Theorem Suppose a curve C is given by parametrized form $(x(t), y(t))$ where $\alpha \le t \le \beta$ and C is traversed exactly once as t increases α to β. Then the length of C is

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Proof idea

Examples

1. Perimeter of a circle

2. Suppose $y = f(x)$ $a \le x \le b$.
   Parametrize by $(t, f(t))$ $a \le x \le b$

$$L = \int_a^b \sqrt{1^2 + f'(t)^2}\, dt$$ Get old formula back!

Area under Curve If $y = y(x) \geq 0$ $a \leq x \leq b$ then

$$\text{Area} = \int_a^b y\,dx$$

EX Area under cycloid $x = r(\theta - \sin\theta)$, $y = r(1-\cos\theta)$

for one arch $(0 \leq \theta \leq 2\pi)$

$$dx = r(1-\cos\theta)\,d\theta$$

$$\int_0^{2\pi} r(1-\cos\theta)\,r(1-\cos\theta)\,d\theta$$

$$= r^2 \int_0^{2\pi} (1 - 2\cos\theta + \cos^2\theta)$$

$$= r^2 \int_0^{2\pi} 1 - 2\cos\theta + \tfrac{1}{2} + \tfrac{1}{2}\cos 2\theta$$

$$= r^2 \int_0^{2\pi} \tfrac{3}{2} - \quad = r^2\left(\tfrac{3}{2}\theta - 2\sin\theta + \tfrac{1}{4}\sin 2\theta\right)\Big|_0^{2\pi}$$

$$= r^2(3\pi - 0) - r^2 0$$

$$= 3\pi r^2$$

3 times area of circle