p. 510 # 1, 2, 8, 12, 19, 34, 35

3/22/07

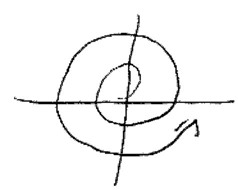
Recall

Suppose we have a polar curve $r = f(\theta)$.
Then we can parameterize it as

$$x = f(\theta)\cos\theta \quad y = f(\theta)\sin\theta, \text{ i.e.}$$

$$(x(\theta), y(\theta)) = (f(\theta)\cos\theta, f(\theta)\sin\theta)$$

Example $r = \theta$



parameterize as $(x,y) = (\theta\cos\theta, \theta\sin\theta)$

Then we can calculate slope of tangent lines as

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta \cancel{+} - f(\theta)\sin\theta}$$

$$= \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

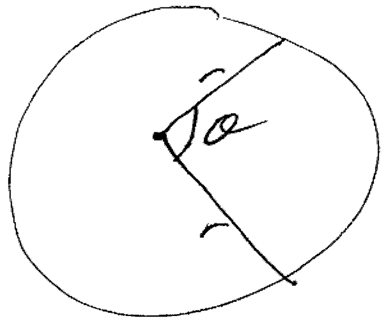
Example Find tangent line to curve $r = \theta$ at point $r = \theta = \pi/4$

A: slope $= \dfrac{\theta\cos\theta + \sin\theta}{-\theta\sin\theta + \cos\theta}$ at $\theta = \pi/4$

$$= \frac{\pi/4 \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}}{-\pi/4 \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}} \quad \text{point } \left(\frac{\pi}{4}\cdot\frac{\sqrt{2}}{2}, \frac{\pi}{4}\frac{\sqrt{2}}{2}\right)$$

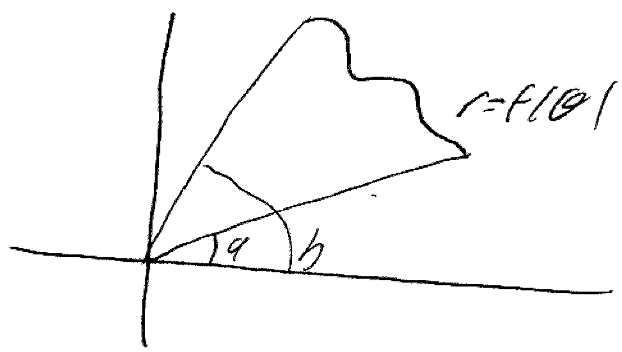
Areas & Arc Length in Polar

Review



$$\text{Area} = \pi r^2 \cdot \frac{\theta}{2\pi} = \frac{1}{2} r^2 \theta$$

Problem Find area enclosed by $r = f(\theta)$ and rays $\theta = a$, $\theta = b$ with $b - a \leq 2\pi$

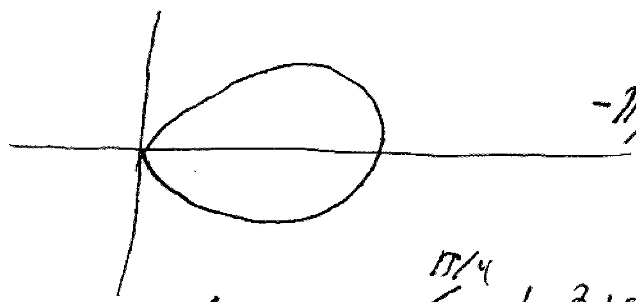


Divide into tiny pieces $\displaystyle\sum_{i=1}^{n} \frac{1}{2} f(\theta_i^*)^2 \Delta\theta$

Theorem The area of this region is $\displaystyle\int_a^b \frac{1}{2}(f(\theta))^2 \, d\theta$

a.k.a. $\displaystyle\int_a^b \frac{1}{2} r^2 \, d\theta$

Example Find area in one leaf of 4-leafed rose $r = |\cos 2\theta|$



$-\pi/4 \le \theta \le \pi/4$

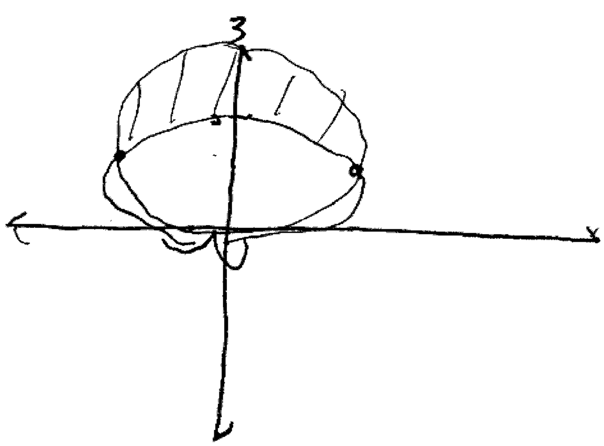
$$\text{Area} = \int_{-\pi/4}^{\pi/4} \tfrac{1}{2} r^2 \, d\theta = \int_{-\pi/4}^{\pi/4} \tfrac{1}{2} \cos^2(2\theta) \, d\theta$$

$$= \frac{1}{2} \int_{-\pi/4}^{\pi/4} \tfrac{1}{2}(1+\cos 4\theta) \, d\theta$$

$$= \frac{1}{4} \left(\theta + \tfrac{1}{4}\sin 4\theta\right)\Big|_{-\pi/4}^{\pi/4}$$

$$= 1/4 \left((\pi/4 + 0) - (-\pi/4 + 0) \right) = \pi/8$$

Example Find area inside circle $r = 3\sin\theta$
outside cardiod $r = 1 + \sin\theta$.



1. Find intersection
$3\sin\theta = 1 + \sin\theta$
$\sin\theta = 1/2 \quad \theta = \pi/6, \; 5\pi/6$

$$\text{Area} = \int_{\pi/6}^{5\pi/6} \tfrac{1}{2} \cdot (3\sin\theta)^2 \, d\theta - \int_{\pi/6}^{5\pi/6} \tfrac{1}{2} (1+\sin\theta)^2 \, d\theta$$

$\vdots$

$= \pi$

Arclength

Problem Find length of $r = f(\theta)$ $a \le \theta \le b$

Solution Parametrize as $(x,y) = (f(\theta)\cos\theta, f(\theta)\sin\theta)$

length =

$$\int_a^b \sqrt{(f'(\theta)\cos\theta - f(\theta)\sin\theta)^2 + (f(\theta)\cos\theta + f'(\theta)\sin\theta)^2}$$

$$= \int_a^b \sqrt{f'(\theta)^2\cos^2\theta - 2f(\theta)f'(\theta)\sin\theta\cos\theta + f(\theta)^2\sin^2\theta + f(\theta)^2\cos^2\theta + 2f(\theta)f'(\theta)\sin\theta\cos\theta + f'(\theta)^2\sin^2\theta}$$

$$= \int_a^b \sqrt{f'(\theta)^2(\sin^2+\cos^2) + f(\theta)^2(\sin^2+\cos^2)}$$

$$= \int_a^b \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}\, d\theta$$

Example Find length of cardiod $r = 1+\sin\theta$

$$\int_0^{2\pi} \sqrt{(1+\sin\theta)^2 + \cos^2\theta}\, d\theta = \int_0^{2\pi} \sqrt{2+2\sin\theta}\, d\theta$$

$$= 8 \quad \text{maple!}$$