p. 429 # 3, 4, 6, 8, 27, 28, 31, 32

3/24/07

Review

Monotonic Sequence Thm

1. Any increasing sequence bounded above converges
2. Any decreasing sequence bounded below converges

"Bounded monotonic sequences must converge"

Rmk This "theorem" is actual an axiomatic property of the real #'s

Ex 3, 3.1, 3.14, 3.141, 3.1415, 3.14159...

is a sequence of rational #s monotonic increasing and bounded above, with no limit in rationals

Example

Let $a_1 = \sqrt{3}$ $a_{n+1} = \sqrt{3 + a_n}$

Claim $a_n < 3$. Proof by induction.

Claim a_n is increasing

Proof by induction

$a_n = \sqrt{3 + a_{n-1}}$

$a_{n-1} = \sqrt{3 + a_{n-2}}$ and $a_{n-1} > a_{n-2}$.

What is the limit?

$x = \sqrt{3 + \sqrt{3 + \sqrt{3 + \cdots}}}$ $x^2 - 3 = x$ $x^2 - x - 3 = 0$ $x = \frac{1 \pm \sqrt{13}}{2}$

limit is $\frac{1 + \sqrt{13}}{2} = 2.30277\ldots$

SERIES

Def An infinite series arises when we add the terms of an infinite sequence,

$$a_1 + a_2 + a_3 + \dots = \sum_{n=1}^{\infty} a_n.$$

Problem What does this mean?

Example $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots = \sum_{n=1}^{\infty} \frac{1}{2^n}$

$$\frac{1}{2} = \frac{1}{2}$$
$$\frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$
$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} = 7/8$$
$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = 15/16$$
$$\frac{1}{2} + \dots + \frac{1}{32} = 31/32$$

$$a_1 = s_1 = 1/2$$
$$a_1 + a_2 = s_2 = 3/4$$
$$a_1 + a_2 + a_3 = s_3 = 7/8$$
$$a_1 + a_2 + a_3 + a_4 = s_4 = 15/16$$

Definition Consider a series $\sum_{n=1}^{\infty} a_n$. Define the n^{th} partial sum

$$s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + \dots + a_n$$

Def Given a Series $\sum_{n=1}^{\infty} a_n$, let s_n be the n^{th} partial sum.

If the sequence $\{s_n\}$ converges to s then we say the series $\sum a_n$ is convergent and write

$$\sum_{n=1}^{\infty} a_n = s$$

Otherwise say the series is divergent.

Logical Order

1. Define Sequences
2. Define limits of sequences
3. Define Series
4. Given a series, obtain a sequence of partial sums
5. Use this sequence to decide if series converges

Example $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$ $s_1 = 1/2$ $s_2 = 3/4$, $s_3 = 7/8$, $s_4 = 15/16$

$$s_n = \frac{2^n - 1}{2^n} \qquad \lim_{n \to \infty} s^n = 1$$

Thus

$$\sum_{i=1}^{\infty} \frac{1}{2^i} = \frac{1}{2} + \frac{1}{4} + \ldots = 1.$$

Geometric Series

Def A geometric series is of the form

$$a + ar + ar^2 + ar^3 + \dots = \sum_{n=1}^{\infty} ar^{n-1}$$

r = ratio a = 1st term

If $r=1$ then $\sum_{n=1}^{\infty} a$ diverges

$$\begin{aligned} S_n &= a + ar + ar^2 + \dots + ar^{n-1} \\ rS_n &= \quad\; ar + ar^2 + \dots + ar^{n-1} + ar^n \end{aligned}$$

$$S_n - rS_n = a - ar^n$$

$$S_n = \frac{a(1-r^n)}{1-r}$$

$$\lim_{n\to\infty} S_n = \begin{cases} \dfrac{a}{1-r} & |r|<1 \\ \text{diverges} & \text{else} \end{cases}$$

Thm $\sum_{n=1}^{\infty} ar^{n-1}$ converges if and only if $|r|<1$,

in which case

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$$

Ex $\frac{1}{2} + \frac{1}{4} + \dots$ $a = \frac{1}{2}$ $r = \frac{1}{2}$ sum $= 1$