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4/3/07

Recall:

Integral Test $f(x)$ continuous, positive & decreasing on $[1,\infty)$
Let $a_n = f(n)$. Then

1. If $\int_1^\infty f(x)\,dx$ converges, so does $\sum_{n=1}^\infty a_n$
2. If $\int_1^\infty f(x)\,dx$ diverges so does $\sum_{n=1}^\infty a_n$

Rmks 1. Can start past 1.
2. Works as long as $f(x)$ decreasing eventually.
3. The decreasing hypothesis lets us use $f(x)$ as upper or lower bound on $\sum a_n$

Example $\sum_{n=1}^\infty \frac{\ln n}{n}$

Let $f(x) = \frac{\ln x}{x}$ This is ≥ 0 and continuous on $[1,\infty)$

$f'(x) = \frac{1-\ln x}{x^2}$ is <0 for $x>e$, Thus decreases eventually.

$\int_1^\infty \frac{\ln x}{x}\,dx = \int u\,du = \frac{1}{2}(\ln x)^2\Big|_1^\infty \to \infty$.
$u = \ln x$

Thus $\boxed{\sum_{n=1}^\infty \frac{\ln n}{n} \text{ diverges}}$

<u>Example</u> When does the <u>p-series</u> $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converge?

Recall: $\int_1^{\infty} \frac{1}{x^p}\,dx$ converges iff $p>1$

Answer: $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges $p>1$, diverges $p \le 1$

A more obvious test..

<u>Comparison Test</u> Let $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$ be series of positive terms.

1. Suppose $a_n \le b_n$ eventually and $\sum_{n=1}^{\infty} b_n$ converges. Then so does $\sum_{n=1}^{\infty} a_n$

2. Suppose $a_n \le b_n$ eventually and $\sum_{n=1}^{\infty} a_n$ diverges. Then so does $\sum_{n=1}^{\infty} b_n$

<u>Proof</u> These are positive series so only question is if partial sums are bounded

Ex We know $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. Now $\frac{1}{n} < \frac{\ln n}{n}$ for $n>3$

Thus $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ diverges by comparison test.

Ex $\sum_{n=1}^{\infty} \frac{6}{3n^2+2n+5}$ $\qquad \frac{6}{3n^2+2n+5} \le \frac{6}{3n^2} = \frac{2}{n^2}$

and $\sum_{n=1}^{\infty} \frac{2}{n^2} = 2 \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges

Thus $\sum_{n=1}^{\infty} \frac{6}{3n^2+2n+5}$ converges

Problem Does $\sum_{n=1}^{\infty} \frac{1}{7^n-5}$ converge?

$\sum \frac{1}{7^n}$ does but $\frac{1}{7^n} < \frac{1}{7^n-5}$ so comparison test no use.

Still it seems as $n \to \infty$ that $\frac{1}{7^n-5} \approx \frac{1}{7^n}$ so should converge.

Thm (Limit comparison Test)

Suppose $\sum a_n$, $\sum b_n$ series of positive terms

If $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$, c finite. Then both series converge or both diverge.

Example

$$\lim_{n\to\infty} \frac{\frac{1}{7^n-5}}{\frac{1}{7^n}} = \lim_{n\to\infty} \frac{7^n}{7^n-5} = \lim_{n\to\infty} \frac{1}{1-\frac{5}{7^n}} = 1$$

But $\sum_{n=1}^{\infty} \frac{1}{7^n}$ converges, so then does $\sum \frac{1}{7^n-5}$

(**) Warning: The 3 tests in this section are for series of positive terms.

Next section gives tests for other series.

<u>Def:</u> An <u>alternating series</u> is of the form $\sum_{n=1}^{\infty} a_n$ where a_n's alternate $+ - + - \ldots$

Ex $\sum_{n=1}^{\infty} \frac{(-1)^n}{n} = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} \ldots$

$\sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{n}{n+3} = \frac{1}{4} - \frac{2}{5} + \frac{3}{6} \ldots$

Alternating Series test

Suppose an alternating series

$b_1 - b_2 + b_3 - b_4 \cdots$ w/ $b_i > 0$.

If $b_{n+1} \leq b_n \; \forall n$ and $\lim_{n\to\infty} = 0$

Then $\sum_{n=1}^{\infty} (-1)^{n-1} b_n$ converges

<u>Proof</u> $S_2 = b_2 - b_1 \geq 0$

$S_4 = S_2 + b_4 - b_3 \geq S_2$ etc. $0 \leq S_2 \leq S_4 \leq \ldots$

But $S_{2n} = b_1 - (b_2 - b_3) - (b_4 - b_5) \cdots - (b_{2n-2} - b_{2n-1}) - b_{2n}$

Thus $S_{2n} \leq b_1$. Thus $\{S_{2n}\}$ converges, say $\lim_{n\to\infty} S_{2n} = S$

Now $S_{2n+1} = S_{2n} + b_{2n+1}$

$\lim_{n\to\infty} S_{2n+1} = S + \lim_{n\to\infty} b_{2n+1} = S$

Thus odd & even partials sums converge to same S