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4/5/07

Review Alternating Series test

Suppose $0 \le b_{n+1} \le b_n$ for all n and $\lim_{n\to\infty} b_n = 0$.

Then $\sum_{n=1}^{\infty} (-1)^n b_n$, $\sum_{n=1}^{\infty} (-1)^{n+1} b_n$ converge.

Proof Remember we must show the sequence of partial sums converges. We do the $(-1)^{n+1}$ case.

$$S_2 = b_1 - b_2 \ge 0$$
$$S_4 = S_2 + b_3 - b_4 \ge S_2 \quad \text{etc.}$$

Thus $S_{2n} = S_{2n-2} + b_{2n-1} - b_{2n} \ge S_{2n-2}$, i.e.

$$0 \le S_2 \le S_4 \le S_6 \ldots \qquad \text{However}$$

$$S_{2n} = b_1 - b_2 + b_3 \ldots \qquad + b_{2n-1} - b_{2n}$$
$$= b_1 - (b_2 - b_3) - (b_4 - b_5) \ldots - (b_{2n-2} - b_{2n-1}) - b_{2n} \le b_1$$

Thus $\{S_2, S_4, S_6, \ldots\}$ monotone $\uparrow$, bounded above so

$$\lim_{n\to\infty} S_{2n} = S$$

But $\lim_{n\to\infty} S_{2n+1} = \lim_{n\to\infty}(S_{2n} + b_{2n+1}) = \lim_{n\to\infty} S_{2n} + \lim_{n\to\infty} b_{2n+1}$
$$= S + 0 = S$$

Thus $\lim_{n\to\infty} S_n = S$. //

RMK Only need b_n's decreasing eventually

Example
Test the series $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n^2+5}{3n^2-1}$

A: This is alternating but $\lim_{n\to\infty} b_n = 1/3 \neq 0$ A.S.T. Does not apply.
Notice the sequence must diverge

Example Test $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n^2}{n^3+1}$

A: $\displaystyle\lim_{n\to\infty} \frac{n^2}{n^3+1} = \lim_{n\to\infty} \frac{\frac{1}{n}}{1+1/n^3} = 0.$

Thus we need b_n's decreasing.

$$f(x) = \frac{x^2}{x^3+1} \qquad f'(x) = \frac{2x(x^3+1) - 3x^4}{(x^3+1)^2}$$
$$= \frac{x(2x^3+2-3x^3)}{(\ )^2} = \frac{x(2-x^3)}{(\ )^2}$$

This is <0 for $x > \sqrt[3]{2}$.

Thus series converges by AST.

Theorem (Alt Series Error Estimate)
Let $\sum_{n=1}^{\infty} (-1)^{n+1} b_n$ be as in A.S.T., so
$$\sum_{n=1}^{\infty} (-1)^{n+1} b_n = S. \quad \text{Then}$$

$$|S - S_n| \leq b_{n+1}$$

Example Find $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3+1}$ correct to 3 decimal places.

Absolute vs Conditional Convergence

Def A series $\sum a_n$ is <u>absolutely convergent</u> if $\sum |a_n|$ converges.

It is <u>conditionally convergent</u> if $\sum a_n$ converges and $\sum |a_n|$ diverges

Examples

1. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ is absolutely convergent.

2. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ is conditionally convergent.

$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} \dots$ converges by AST

$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$ diverges, harmonic series

Thm Absolute convergent $\longrightarrow$ convergent

<u>Proof</u> Suppose $\sum |a_n|$ converges

$$0 \le a_n + |a_n| \le 2|a_n|$$

Thus $\sum a_n + |a_n|$ converges by Comparison Test

And $\sum |a_n|$ converges

So $\sum a_n + |a_n| - \sum |a_n| = \sum a_n$ converges

<u>Example</u>

$$1 - \frac{1}{4} - \frac{1}{9} + \frac{1}{16} - \frac{1}{25} - \frac{1}{36} + \frac{1}{49} - \frac{1}{64} - \frac{1}{81} \ldots$$

converges since
it converges absolutely.

<u>Example</u>

Does $\sum_{n=1}^{\infty} \frac{\sin n}{n^3}$ converge?

$0 \le \left|\frac{\sin n}{n^3}\right| \le \frac{1}{n^3}$ so $\sum \left|\frac{\sin n}{n^3}\right|$ converges

so $\sum \frac{\sin n}{n^3}$ does