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p. 471 1, 2, 3, 6, 9,
12, 14, 21

Suppose $f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 \ldots$

$= \sum_{n=0}^{\infty} c_n (x-a)^n$

Notice $\quad c_0 = f(a)$

$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3$

$c_1 = f'(a)$

$f''(x) = 2c_2 + 3 \cdot 2 c_3(x-a) + 4 \cdot 3 c_4 (x-a)^2 \ldots$

$c_2 = \frac{1}{2} f''(a)$

$f'''(x) = 3 \cdot 2 \cdot 1 c_3 + 4 \cdot 3 \cdot 2 (x-a) + 5 \cdot 4 \cdot 3 (x-a)^2 + \ldots$

$c_3 = \frac{1}{3 \cdot 2 \cdot 1} f'''(a)$

<u>Theorem</u> Suppose $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ holds for $|x-a| < R$

Then $c_n = \frac{f^{(n)}(a)}{n!}$

i.e. $f(x) = f(a) + \frac{f'(a)}{1}(x-a) + \frac{f''(a)}{2 \cdot 1}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 \ldots$

<u>Def</u> $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$ is the <u>Taylor series</u> of $f(x)$ centred at a.

When $a = 0$, sometimes called Maclaurin Series

<u>Warning</u> If $f(x)$ can be represented as a power series near a this theorem gives the coefficients. It does not guarantee that $f(x)$ has a power series near.

<u>Example</u> $f(x) = e^x$ $\quad f(0)=1 \quad f'(0)=1 \quad f''(0)=1 \ldots$

The Taylor (Maclaurin) series is $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$

* <u>If</u> e^x has a power series, this is it *

<u>Def</u> $T_n(x) = f(a) + \dfrac{f'(a)}{1!}(x-a) + \dfrac{f''(a)}{2!}(x-a)^2 + \cdots + \dfrac{f^{(n)}(a)}{n!}(x-a)^n$

is the n^{th} degree Taylor Polynomial of f at a.

Ex $f(x) = e^x \quad a = 0$

$T_1(x) = 1+x \quad T_2(x) = 1+x+\dfrac{x^2}{2} \quad T_3(x) = 1+x+\dfrac{x^2}{2}+\dfrac{x^3}{6}$

<u>Def</u> $R_n(x) = f(x) - T_n(x)$ <u>remainder</u>. If $\lim_{n\to\infty} R_n(x) = 0$

then $\lim_{n\to\infty} T_n(x) = f(x)$ as we want.

<u>Thm</u> If $f(x) = T_n(x) + R_n(x)$ and $\lim_{n\to\infty} R_n(x) = 0$ for $|x-a| < R$ then $f(x)$ is equal to its Taylor Series on $|x-a| < R$.

Q How to show $\lim_{n\to\infty} R_n(x) = 0$?

<u>Taylor's Formula</u> Suppose $f(x)$ has $n+1$ derivatives on an interval I containing a. Then for $x \in I$ there is z between x & a such that

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-a)^{n+1}$$

<u>Example</u> $n=0$ $x=b$ $z=c$ $f(b) = f(a) + f'(c)(b-a)$

MVT.

<u>Useful Fact</u> $\lim_{n\to\infty} \frac{x^n}{n!} = 0$ for any x.

<u>Example</u>

Prove e^x is equal to the sum of its Taylor Series.

<u>Proof</u> $R_n(x) = \frac{e^z}{(n+1)!} x^{n+1}$ for z between 0 & x.

If $x>0$ then $0 < e^z < e^x$ so

$$0 < R_n(x) < e^x \cdot \frac{x^{n+1}}{(n+1)!} \to 0 \text{ for all } x.$$

Similar for $x<0$.

Thus $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for any x.

<u>Example</u> $f(x) = \sin x$. Find Maclaurin series. Show it represents $\sin x$ for all x.

$f(x) = \sin x$ $\quad f'(x) = \cos x$ $\quad f''(x) = -\sin x$ $\quad f'''(x) = -\cos x$

$f^{(4)}$ $\quad f^{(5)}$ ..

Thus $f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \cdots$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!} x^n$$

But $0 \le |f^{(n+1)}(z)| \le 1$ for any n or z.

Thus
$$0 \le R_n(x) \le \frac{|x|^n}{(n+1)!} \to 0,$$

<u>COR</u> $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \cdots$

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \cdots$