4/24/07 p.471 #27, 32, 43, 54, 58, 63, 64

Review Given $f(x)$. The Taylor series for $f(x)$ centered at $x=a$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Problem When is it equal to $f(x)$?

Def $T_n(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n$

is n^{th} degree Taylor polynomial of $f(x)$ at $x=a$

Ex $f(x) = e^x$, $a=0$ $\quad T_1(x) = 1+x \quad T_2(x) = 1+x+\frac{x^2}{2} \quad T_3(x) = 1+x+\frac{x^2}{2}+\frac{x^3}{6}$

Def $R_n(x) = f(x) - T_n(x)$ remainder.

If $\lim_{n\to\infty} R_n(x) = 0$ then $\lim_{n\to\infty} T_n(x) = f(x)$ as we want.

Thm If $f(x) = T_n(x) + R_n(x)$ and $\lim_{n\to\infty} R_n(x) = 0$ for $|x-a| < R$
Then $f(x)$ is equal to its Taylor series on $|x-a| < R$

Problem We need to prove $\lim_{n\to\infty} R_n(x) = 0$.

Taylors Formula

Suppose $f(x)$ has $n+1$ derivatives on an interval I containing a. Then for any $x \in I$ there is a z between x and a such that

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-a)^{n+1}$$

Example. Prove e^x is equal to the sum of its Taylor series

Proof Let $a=0$. By Taylor's Formula

$$R_n(x) = \frac{e^z}{(n+1)!} x^{n+1} \quad \text{for some } z \text{ between } 0 \text{ and } x$$

If $x>0$ then $0 < e^z < e^x$ so

$$0 < R_n(x) < e^x \cdot \frac{x^{n+1}}{(n+1)!}$$

But $\lim_{n\to\infty} \frac{x^n}{n!} = 0$ for any x

Thus $\lim_{n\to\infty} R_n(x) = 0$. Same for $x<0$

Thus

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad x \in (-\infty, \infty)$$

Example Consider Taylor's Formula $n=0$, $x=b$, $z=c$.

Then

$$\frac{f'(c)}{1}(b-a) = f(b) - f(a)$$

MVT!

Example

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \quad x \in (-\infty, \infty)$$

Proof

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!} x^n$$

$$0 \le |R_n(x)| \le \frac{|x|^n}{(n+1)!} \to 0.$$

Thm You can multiply power series
If both have radius of conv. R, so does the product

You can divide if denom has $\neq 0$ constant term
still converges but R may be smaller.

Problems

1. $f(x) = 1/\sqrt{x}$ $a = 9$ Find T.S Do not show $R_n(x) \to 0$
2. $f(x) = \ln x$ $a = -1$ " " " "

3). Use Maclaurin series from section to find
$f(x) = x^2 e^{-x}$

Find $e^{-.2}$ correct to 5 decimal places.

Find First 3 nonzero terms in
Macl series, $y = e^{-x^2} \cos x$

, $y = \dfrac{x}{\sin x}$