Name: SOLUTIONS

Math 1840- Midterm Exam #2 - March 16, 2007

1. **(15 points)** Evaluate the following limits.

$a.\ \lim_{x\to 0}(1-3x)^{1/x}$ $\qquad b.\ \lim_{x\to \pi/2^-}\frac{1-\sin x}{\cos x}$ $\qquad c.\ \lim_{x\to \pi/2}\frac{\sin x}{1+\cos x}$

a. $y = \lim_{x\to 0}(1-3x)^{1/x}$

$\ln y = \lim_{x\to 0}\frac{\ln(1-3x)}{x} \overset{L.H.R.}{=} \lim_{x\to 0}\frac{\frac{-3}{1-3x}}{1} = -3$

So $y = e^{-3}$

b. $\overset{LHR}{=} \lim_{x\to \pi/2^-}\frac{-\cos x}{-\sin x} = \frac{0}{1} = 0$

c. $\lim_{x\to \pi/2}\frac{\sin x}{1+\cos x} = \frac{1}{1} = 1$ LHR does not apply

Trigonometric identities:

$$\sin^2 x + \cos^2 x = 1, \quad \tan^2 x + 1 = \sec^2 x, \quad 1 + \cot^2 x = \csc^2 x$$

$$\sin(2x) = 2\sin x \cos x, \quad \sin^2 x = (1 - \cos(2x))/2, \quad \cos^2 x = (1 + \cos(2x))/2.$$

$$\int \sec x \, dx = \ln|\sec x + \tan x| + C \qquad \int \tan x \, dx = \ln|\sec x| + C.$$

2. **(15 points)** Find the arc length of the curve

$$y = \ln(\cos x) \;\; 0 \le x \le \pi/3.$$

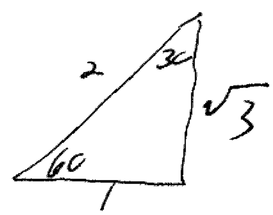
$$y' = \frac{-\sin x}{\cos x} = -\tan x$$

$$A.L. = \int_0^{\pi/3} \sqrt{1+\tan^2 x}\, dx$$

$$\cancel{u = \tan x \quad du = \sec^2 x}$$

$$= \int_0^{\pi/3} \sec x \, dx$$

$$= \ln|\sec x + \tan x| \Big|_0^{\pi/3}$$



$$\sec(\pi/3) = 2$$
$$\tan(\pi/3) = \sqrt{3}$$
$$\sec 0 = 1$$
$$\tan 0 = 0$$

$$= \ln|2+\sqrt{3}| - \ln|1|$$

$$= \ln|2+\sqrt{3}|$$

3. **(20 points)**

a. $\int \sec^5 x \tan x dx$

b. $\int_0^{\pi} x \cos x dx$

a. $u = \sec x \quad du = \sec x \tan x \, dx$

$= \int u^4 du = \frac{1}{5} u^5 + C = \boxed{\frac{1}{5} \sec^5 x + C}$

b. $u = x \qquad v = \sin x$

$du = dx \qquad dv = \cos x \, dx$

$\int x \cos x \, dx = x \sin x - \int \sin x \, dx$

$= x \sin x + \cos x$

$x \sin x + \cos x \Big|_0^{\pi} = (0 + -1) - (0 + 1)$

$= -2$

4. **(30 points)**

a. $\int_2^\infty \frac{1}{x^3}dx$ b. $\int \sin^2 x\, dx$ c. $\int \frac{x+1}{x^2+6x+14}dx$

a. $= \lim_{t\to\infty} \left(-\frac{1}{2}x^{-2}\right)\Big|_2^t = \lim_{t\to\infty}\left(\frac{-1}{2x^2}\right)\Big|_2^t$

$= \lim_{t\to\infty}\left(\frac{-1}{2t^2} + \frac{1}{8}\right) = 1/8$

b. $= \int \frac{1}{2} - \frac{1}{2}\cos 2x\, dx = \boxed{\frac{1}{2}x - \frac{1}{4}\sin 2x + C}$

c. $x^2+6x+14 = (x+3)^2+5$ $u = x+3$ $x = u-3$

$du = dx$

$= \int \frac{u-2}{u^2+5}du = \int \frac{u}{u^2+5}du - 2\int \frac{1}{u^2+5}du$

$= \frac{1}{2}\ln|u^2+5| - \frac{2}{\sqrt{5}}\tan^{-1}\left(\frac{u}{\sqrt{5}}\right) + C$

$= \frac{1}{2}\ln|x^2+6x+14| - \frac{2}{\sqrt{5}}\tan^{-1}\left(\frac{x+3}{\sqrt{5}}\right) + C$

5. **(10 points)** Consider the parameterized curve

$$(x(t), y(t)) = (t^2 + 3t, 1 + t + t^2) \ 0 \leq t \leq \pi.$$

a. Find the equation of the tangent line to this curve at the point corresponding to $t = 1$.

$$\vec{r}'(t) = (2t+3, 1+2t)$$

$$r'(1) = (5, 3) \qquad \text{slope} = 3/5$$

$$\text{point} = r(1) = (4, 3)$$

$$y - 3 = 3/5(x - 4)$$

b. Write down but do not evaluate an integral which gives the length of this curve.

$$\int_0^{\pi} \sqrt{(2t+3)^2 + (1+2t)^2}\, dt$$

6. **(10 points)** Sketch the following curve, indicating with an arrow the direction the curve is traced as the parameter increases. Hint: First eliminate the parameter t to get the equation in Cartesian coordinates.

$$(x(t), y(t)) = (2\sin t - 3, 2\cos t + 4) \ 0 \leq t \leq 2\pi$$

$$(x+3)^2 + (y-4)^2 = 4(\sin^2 t + \cos^2 t) = 4$$

Circle, center $(-3, 4)$ radius 2.

$t = 0 \to (-3, 6) \qquad t = \pi/2 \to (-1, 4)$