Name: SOLUTIONS

Math 1840- Final Exam - May 3, 2007

1. **(20 points)** True or false:

T a. If the series $\sum_{n=1}^{\infty} a_n$ converges then $\lim_{n\to\infty} a_n = 0$.

T b. $y = \cosh x$ is an even function.

T c. If f is one-to-one with domain $(-\infty, \infty)$ then $f^{-1}(f(6)) = 6$.

T d. 5^x is positive for any x in $(-\infty, \infty)$.

F e. The inverse function of $y = e^{5x}$ is $y = \ln(\frac{x}{5})$

F f. The alternating series test can be used to prove a series converges absolutely.

T g. The ratio test can be used to determine whether $\sum \frac{1}{n!}$ converges.

T h. The series $\sum_{n=0}^{\infty} \frac{(-1)^n}{n\sqrt{n}}$ converges absolutely.

F i. $\frac{\ln a}{\ln b} = \ln a - \ln b$

T j. $e^{2\ln x} = x^2$

2. **(10 points)** Use a power series to estimate $1/\sqrt{e}$ correct to within 1/200. You may use your calculator to perform *arithmetic* only, i.e. not to take square roots or enter "e". Be sure to explain how you know your estimate is accurate enough. Give your final answer as a *fraction* in lowest terms. Show all your work.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$$

$$\frac{1}{\sqrt{e}} = e^{-1/2} = 1 - \frac{1}{2} + \frac{1/4}{2} - \frac{1/8}{6} + \frac{1/16}{24}$$

$$= 1 - 1/2 + 1/8 - 1/48 + 1/388 \ldots$$

By Alt Series error estimate, $1 - 1/2 + 1/8 - 1/48$ is accurate w/ error at most $1/388$ which is $< 1/200$.

$$1/\sqrt{e} \approx 1 - 1/2 + 1/8 - 1/48$$

$$= 29/48$$

3. **(10 points)** Find the radius of convergence and the interval of convergence for each of the series below:

$$a.\ \sum_{n=0}^{\infty} \frac{2^n(x-3)^n}{\sqrt{n+3}} \qquad\qquad b.\ \sum_{n=0}^{\infty}(-1)^n \frac{n^2}{(n+1)!}x^n$$

a. $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{2^{n+1}(x-3)^{n+1}}{\sqrt{n+4}}}{\frac{2^n(x-3)^n}{\sqrt{n+3}}}\right| = \left|\frac{\sqrt{n+3}}{\sqrt{n+4}}\, 2|x-3|\right|$

So $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 2|x-3|$

$-1 < 2x-6 < 1$

$5 < 2x < 7$

$5/2 < x < 7/2$ $\quad R = 1/2$

$x = 5/2 \Rightarrow \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+3}}$ converges AST

$x = 7/2 \Rightarrow \sum \frac{1}{\sqrt{n+3}}$ diverges.

$[5/2, 7/2)$

b. $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{(n+1)^2}{(n+2)!}x^{n+1}}{\frac{n^2}{(n+1)!}x^n}\right| = \left|\left(\frac{n+1}{n}\right)^2 \cdot \frac{x}{n+2}\right|$

$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 0$ for any x.

$R = \infty$

$(-\infty, \infty)$

4. **(10 points)** Evaluate the indefinite integral as an infinite series:

$$\int \frac{\cos(x^2) - 1}{x} dx$$

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4} \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$\cos(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{(2n)!}$$

$$\cos(x^2) - 1 = \sum_{n=1}^{\infty} \frac{(-1)^n x^{4n}}{(2n)!}$$

$$\frac{\cos(x^2) - 1}{x} = \sum_{n=1}^{\infty} \frac{(-1)^n x^{4n-1}}{(2n)!}$$

$$\int \frac{\cos(x^2) - 1}{x} dx = \sum_{n=1}^{\infty} \frac{(-1)^n x^{4n}}{4n \cdot (2n)!}$$

5. **(10 points)** Find the Taylor polynomial of degree 2 for $f(x) = \sqrt{1+x}$ centered at $a = 3$.

$f(x) = \sqrt{1+x}$ $\quad f(3) = 2$

$f'(x) = \frac{1}{2\sqrt{1+x}}$ $\quad f'(3) = 1/4$

$= \frac{1}{2}(1+x)^{-1/2}$

$f''(x) = -\frac{1}{4}(1+x)^{-3/2} = \frac{-1}{4\sqrt{1+x}^3}$ $\quad f''(3) = -1/32$

Taylor poly is $f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2$

$= 2 + \frac{1}{4}(x-3) - \frac{1}{64}(x-3)^2$

6. **(10 points)** A 100° object is brought into a room whose temperature is kept at a constant 60°. The object cools from 100° to 85° in 30 minutes. How *much longer* will it take for the object to reach 70°.

Let $y(t)$ = difference between object and 60°.

So $y(t) = y_0 e^{kt}$. Given $y(0) = 40$ $y(30) = 25$

$y(t) = 40e^{kt}$

$25 = 40e^{30k}$

$\frac{5}{8} = e^{30k}$ $\quad k = \frac{\ln(5/8)}{30}$

$$y(t) = 40e^{\frac{\ln(5/8)}{30}t}$$

Want $y(t) = 20$!

$$20 = 40e^{\frac{\ln(5/8)}{30}t}$$

$$\frac{1}{2} = e^{\frac{\ln(5/8)}{30}t}$$

$$\ln(1/2) = t \cdot \ln(5/8)/30$$

$$t = \frac{30\ln(1/2)}{\ln(5/8)}$$

7. **(5 points)** A curve passes through the point $(0, 2)$ and has the property that the slope of the curve at every point P is three times the y coordinate at P. What is the equation of the curve?

$$y = 2e^{3x}$$

8. **(12 points)** Find $\frac{dy}{dx}$:

a. $y = 10^x$

b. $y = \cosh x$

c. $y = \log_7 x$

d. $y = \frac{(x^2+5)^3(x+3)^6}{(e^x+5)^8}$

a. $\boxed{10^x \ln 10}$ b. $\boxed{\sinh x}$ c. $\boxed{\frac{1}{x \ln 7}}$

d. $\ln y = 3\ln(x^2+5) + 6\ln(x+3) - 8\ln(e^x+5)$

$$\frac{1}{y}y' = \frac{6x}{x^2+5} + \frac{6}{x+3} - \frac{8e^x}{e^x+5}$$

$$y' = \frac{(x^2+5)^3(x+3)^6}{(e^x+5)^8}\left(\frac{6x}{x^2+5} + \frac{6}{x+3} - \frac{8e^x}{e^x+5}\right)$$

9. **(10 points)** Evaluate the following limits:

a. $\lim_{x\to(\pi/2)^-}(\sec x - \tan x)$.

b. $\lim_{x\to 0^+} x^{x^2}$

a. $\lim\left(\frac{1}{\cos x} - \frac{\sin x}{\cos x}\right) = \lim_{x\to\pi/2^-}\left(\frac{1-\sin x}{\cos x}\right) \overset{LHR}{=} \lim_{x\to\pi/2^-} \frac{-\cos x}{-\sin x}$

$= 0$

b. $y = x^{x^2}$ $\quad \ln y = x^2 \ln x$

$\lim_{x\to 0} x^2 \ln x = \lim_{x\to 0} \frac{\ln x}{1/x^2} \overset{LHR}{=} \lim_{x\to 0} \frac{1/x}{-2/x^3}$

$= \lim_{x\to 0} \frac{x^2}{2} = 0$

so $y \to e^0 = 1$

10. **(12 points)** Do the following integrals:

a. $\int \frac{x}{(x-1)^2(x^2+2x+4)}dx$

b. $\int \frac{6}{x-5}dx$

c. $\int \sec^4 x \tan x dx$

a. Assume it is $\int \frac{1}{x-1} + \frac{1}{(x-1)^2} + \frac{x+1}{x^2+2x+4}\,dx$ (announced during final)

$= \boxed{\ln|x-1| - \frac{1}{x-1} + \frac{1}{2}\ln|x^2+2x+4| + C}$

b. $\boxed{6\ln|x-5| + C}$

c. $u = \sec x \quad du = \sec x \tan x\, dx$

$\int u^3 du = \frac{u^4}{4} = \boxed{\frac{\sec^4 x}{4} + C}$

11. **(12 points)** Do the following integrals:

a. $\int \sqrt{1-x^2}dx$

b. $\int x\cos x dx$

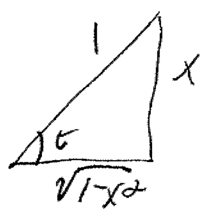
c. $\int_1^3 \frac{1}{\sqrt[3]{x-1}}$

a. $x=\sin t \quad dx=\cos t\, dx \quad \int\sqrt{1-\sin^2 t}\,\cos t\, dt$

$= \int \cos^2 t\, dt = \int \frac{1}{2}+\frac{\cos 2t}{2}$

$= \frac{1}{2}t + \frac{\sin 2t}{4}$

$t=\sin^{-1}x$



$\sin 2t = 2\sin t\cos t$

$= 2x\cdot\sqrt{1-x^2}$

$$\frac{\sin^{-1}x}{2}+\frac{2x\sqrt{1-x^2}}{4}+C$$

b. $x\sin x + \cos x + c$

c. $\int_1^3 (x-1)^{-1/3}dx = \frac{(x-1)^{2/3}}{2/3}\Big|_1^3 = \frac{3}{2}(2^{2/3}-0)$

$= \frac{3}{\sqrt[3]{2}}$

12. **(10 points)** Find the equation of the tangent line to the curve

$$(x(t), y(t)) = (\ln t, 1 + t^2)$$

at the time $t = 1$.

$v(t) = \langle \frac{1}{t}, 2t \rangle$ slope $= \frac{2}{1} = 2$

point $= (0, 2)$

$\boxed{y - 2 = 2x}$

13. **(10 points)** Find the area enclosed by the polar curve $r = \sin\Theta$ for $0 \leq \Theta \leq \pi$.

$A = \int_0^\pi \frac{1}{2} \sin^2\theta \, d\theta$

$= \int_0^\pi \frac{1}{2}\left(\frac{1}{2} - \frac{\cos 2\theta}{2}\right) d\theta$

$= \frac{1}{4}\theta - \frac{1}{8}\sin 2\theta \Big|_0^\pi$

$= \pi/4$

14. **(9 points)** State precisely the limit comparison test for series. Be sure to include any hypothesis on the series which are required for the test to apply.

Let Σa_n, Σb_n be positive series.

If $\lim_{n \to \infty} \frac{a_n}{b_n} = c > 0$ then

either both converge or both

diverge.