DUE MONDAY 3/5/07

Fourier Polynomials

Let f be a continuous function on the interval $[-\pi, \pi]$. We define the Fourier coefficients of f by the formulae:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx \quad \text{and} \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx, \qquad n = 0, 1, 2, \ldots .$$

For $N = 1, 2, \ldots$, we define the "Fourier polynomials" of f to be

$$P_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N}[a_n\cos(nx) + b_n\sin(nx)].$$

Fourier coefficients and polynomials are named for the French mathematician Joseph Fourier. In a speech for the French Academy of Sciences in 1807, Fourier proposed that these polynomials could be used to approximate arbitrary functions on the interval $[-\pi, \pi]$. He was led to study these polynomials and make his bold statement by studying heat flow.

Unfortunately, the great French mathematicians of the time did not agree and did not take his ideas seriously. They were wrong. Today, the ideas introduced by Fourier are used to study phenomena that exhibit wavelike (*periodic*) behavior, such as sound and light. These ideas have been used in physics, engineering, and even economics. This problem is an introduction to some of Fourier's work.

a. Let $f(x) = x$. Find all of the Fourier coefficients of f. What are the first five Fourier polynomials P_N associated to f? Plot the function f and these five polynomials.

b. Let f be an odd function, continuous on the interval $[-c, c]$. Prove that
$$\int_{-c}^{c} f(x)\,dx = 0.$$

c. In part (a), something special happened because the function, $f(x) = x$, is an odd function on $[-\pi, \pi]$. What pattern did you notice? Next, let f be any odd function on $[-\pi, \pi]$. Make a conjecture about the Fourier coefficients of f. Prove your conjecture.

d. Another special collection of functions on an interval $[-c, c]$ is the even functions. After thinking about what happens for the odd functions, make a conjecture about the Fourier coefficients of an even function. Let $f(x) = x^2$; check your conjecture using this even function. Explain why your conjecture is true for any even function.

Extra credit: Why did Fourier think this would be a good technique for approximating periodic functions?

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