Harmonic Summing

An important constant in mathematics is defined by the limit,

$$\lim_{n\to\infty}\left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\log n\right).$$

In this project we are going to show that this limit exists and estimate its value. Set

$$T_n = \frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\log n.$$

It is convenient to work first with a closely related sequence,

$$S_n = \frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\log(n+1).$$

a. Draw a picture using areas to relate the sum of the reciprocals of the first n positive integers to the values of the logarithmic function. Identify $\lim_{n\to\infty} S_n$ geometrically. How can the area giving S_n be enclosed in a sequence of n rectangles, each of length at most one on any side?

b. Obtain an upper bound on the sequence $\{S_n\}_{n=1}^{\infty}$ by summing the areas of the rectangles observed in part (a). Show that the sequence, $S_1, S_2, S_3, \ldots$ is increasing. From this one can conclude that this sequence has a limit, since any bounded increasing sequence has a limit.

c. Prove that if $\lim_{n\to\infty} S_n$ exists, then so does $\lim_{n\to\infty} T_n$.

Hint: Prove that the sequence, $T_n - S_n$, converges to zero. Conclude that the limit in question does exist.

d. You are now going to get a lower bound on the limit of the S_n. On your graph connect with straight line segments the points on the logarithmic curve with integral x-coordinates. The sum of the areas of the resultant triangles is less than $\lim_{n\to\infty} S_n$. Evaluate this sum of triangular areas.

Extra credit: Compute the first few terms in the sequence S_n. You are now going to find a more efficient way to get closer to the limit. Subdivide the intervals of unit length. There will now be new areas to add to your lower bound. Do so, and simplify. This should lead you to an expression with an infinite sum. Evaluate it and thereby obtain a better lower bound.

Subdivide again and repeat this process. Continue to subdivide. Develop a general expression for an arbitrary estimate in the process. These estimates will converge much faster than the original sequence.

A similar process can be started on the other side of the logarithmic curve to reduce the upper bound. Develop this and give a general expression for an arbitrary step. In this way the constant will be sandwiched between two new sequences that come arbitrarily close together. Incidentally you may want to prove for the original sequence that $T_n > T_{n+1}$, and hence that both sequences converge.

Comments

Requires: harmonic series