9. $f(x) = \sinh x \quad f'(x) = \cosh x \quad f''(x) = \sinh x$ etc.

$\sinh 0 = 0 \quad \cosh 0 = 1$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots = \boxed{\sum_{n=1}^{\infty} \frac{x^{2n-1}}{(2n-1)!}}$$

$$\boxed{R = \infty}$$

12. $f(x) = x^3 \quad a = -1 \qquad f(-1) = -1 \quad f'(-1) = 3 \quad f''(-1) = -6 \quad f'''(-1) = 6$

$f'(x) = 3x^2$ f

$f''(x) = 6x$

$f'''(x) = 6$

$$\boxed{f(x) = -1 + 3(x+1) - 3(x+1)^2 + (x+1)^3}$$

14. $f(x) = \ln x \quad a = 2$

$f'(x) = \frac{1}{x} \quad f'(2) = 1/2$

$f''(x) = -1/x^2 \quad f''(2) = -1/4$

$f'''(x) = 2/x^3 \quad f'''(2) = 2/8$

$f^{(4)}(x) = -32/x^4 \quad f^{(4)}(2) = -32/2^4$

$$f^{(n)}(2) = (-1)^{n+1} \cdot \frac{(n-1)!}{2^n}$$

Thus Taylor Series is

$$f \sum_{n=0}^{\infty} \frac{f^{(n)}(2)}{n!} \cdot (x-2)^n = \boxed{\sum_{n=0}^{\infty} (-1)^{n+1} \cdot \frac{x^n}{n 2^n}}$$

#21 Class Notes 10/24

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1. $f(x) = \sum_{n=0}^{\infty} b_n (x-5)^n \qquad b_8 = \frac{f^{(8)}(5)}{8!}$

2. a. $f(1)$ is not 1.6

b $f'(2)$ is not 0.5, it is clearly > 1.

3 If $f^{(n)}(0) = (n+1)!$ Then

Maclaurin Series is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} (n+1) x^n$$

Ratio Test $\left| \frac{a_{n+1}}{a_n} \right| = \frac{n+2}{n+1} |x| \quad \lim_{n\to\infty}$ is $|x|$

$R = 1$

6 $f(x) = \sin 2x \quad f'(x) = 2\cos 2x \quad f''(x) = -4\sin 2x \quad f'''(x) = -8\cos 2x$

$f^{(4)} = 16 \sin 2x$

M.S. is $\frac{2}{1!}x - \frac{8}{3!}x^3 + \frac{32}{5!}x^5 - \frac{2^7}{7!}x^7$

$$= \sum_{n=1}^{\infty} \frac{2^n (-1)^{n+1}}{\cancel{(2n+1)!}\ (2n-1)!} x^{2n-1}$$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{2^{n+1} x^{2n+1}}{(2n+1)!}}{\frac{2^n x^{2n-1}}{(2n-1)!}} \right| = \lim_{n\to\infty} \left| \frac{2}{(2n+1)(2n)} x^2 \right|$$

$= 0$ for all x.

$R = \infty$

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#27 $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \ldots$ Thus

$$\boxed{\cos(\pi x) = 1 - \frac{\pi^2 x^2}{2!} + \frac{\pi^4 x^4}{4!} - \frac{\pi^6 x^6}{6!} + \frac{\pi^8 x^8}{8!} \ldots}$$

32 $x\cos 2x = x\left(1 - \frac{4x^2}{2!} + \frac{16x^4}{4!} - \frac{64x^6}{6!}\right) = x - \frac{4x^3}{2!} + \frac{16x^5}{4!} \ldots$

$$= \boxed{\sum_{n=1}^{\infty} 2^{n-1} \frac{x^{2n-1}}{(2n-2)!}}$$

43 $\int x\cos(x^3)\,dx = \int x\left(1 - \frac{x^6}{2!} + \frac{x^{12}}{4!} - \frac{x^{18}}{6!} \ldots\right)$

$$= \int x - \frac{x^7}{2!} + \frac{x^{13}}{4!} - \frac{x^{19}}{6!} \ldots$$

$$= \frac{x^2}{2} - \frac{x^8}{8\cdot 2!} + \frac{x^{14}}{14\cdot 4!} - \frac{x^{20}}{20\cdot 6!} \ldots + C$$

$$\boxed{= C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+2}}{(6n+2)(2n)!}}$$

56 $\sec x = 1/\cos x = $

$$1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} \ldots$$

$$1 + \frac{x^2}{2} + \frac{5}{24}x^4$$

$$1 \big/ \left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} \ldots\right)$$

$$\frac{x^2}{2} - \frac{x^4}{24} + \frac{x^6}{720}$$

$$\frac{x^2}{2} - \frac{x^4}{4} + \frac{x^6}{48}$$

$$\frac{5}{24}x^4 \ldots$$

$$\boxed{\sec x = 1 + \frac{x^2}{2} + \frac{5}{24}x^4 + \ldots}$$

58 $y = e^x \ln(1-x)$

e^x

$$= (1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots)(-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} \cdots)$$

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$$= -x - \frac{x^2}{2} - x^2 - \frac{x^3}{3} - \frac{x^3}{2} - \frac{x^3}{2} + \cdots$$

$$= \boxed{-x - \frac{3}{2}x^2 - \frac{4}{3}x^3 + \cdots}$$

63 $3 + \frac{9}{2!} + \frac{27}{3!} + \cdots = \boxed{e^3 - 1}$ simp

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \cdots$

$e^3 = 1 + 3 + \frac{9}{2!} \cdots$

64 $1 - \ln 2 + \frac{(\ln 2)^2}{2!} - \frac{(\ln 2)^3}{3!} \cdots$

$$= e^{-\ln 2} = e^{\ln(1/2)} = 1/2$$

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#23 $\sqrt{1+x} = (1+x)^{1/2}$

$$= \sum_{n=0}^{\infty} \binom{1/2}{n} x^n$$

$$= 1 + \frac{1/2}{1}x + \frac{\frac{1}{2}\cdot -\frac{1}{2}}{2\cdot 1}x^2 + \frac{\frac{1}{2}\cdot -\frac{1}{2}\cdot -\frac{3}{2}}{3\cdot 2\cdot 1}x^3 + \frac{\frac{1}{2}\cdot -\frac{1}{2}\cdot -\frac{3}{2}\cdot -\frac{5}{2}}{4\,3\,2\,1}x^4$$

$$= 1 + \frac{x}{2} - \frac{x^2}{2^3\cdot 2!} + \frac{3\cdot 1 x^3}{2^3\cdot 3!} - \frac{5\cdot 3\cdot 1 x^4}{2^4\cdot 4!}$$

$$= \boxed{1 + \frac{x}{2} + \sum_{n=2}^{\infty} (-1)^{n-1} \frac{1\cdot 3\cdot 5\cdots(2n-3)}{2^n\cdot n!} x^n} \quad R=1$$

24. $\frac{1}{(1+x)^4} = (1+x)^{-4}$

$$= 1 - 4x + \frac{-4\cdot -5}{2!}x^2 + \frac{-4\cdot -5\cdot -6}{3!}x^3 \ldots$$

$$= \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n (n+3)!}{6\cdot n!} x^n}$$

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11. $f(x) = \tan x \quad a=0 \quad n=3 \quad 0 \le x \le \pi/2$

a. $f'(x) = \sec^2 x \quad f''(x) = 2\sec x\cdot \sec x \tan x$

$f'''(x) = 2\sec x \tan x \sec x \tan x + 2\sec x \tan x \sec x \tan x + 2\sec^4 x$

$= 4\sec^2 \tan^2 x + 2\sec^4 x$

$\tan 0 = 0 \quad \sec 0 = 1$

$f(0)=0 \quad f'(0)=1 \quad f''(0)=0 \quad f'''(0)=2$

$T_3(x) = x + \frac{1}{3}x^3$

11 a.

$f(x) = \tan x \quad f'(x) = \sec^2 x \quad f''(x) = 2\sec^2 x \tan x$

$f'''(x) = 4 \sec x \sec x \tan x \tan x + 2\sec^4 x$

$= 4\sec^2 x \tan^2 x + 2\sec^4 x$

$f^{(4)}(x) = 8 \sec x \sec x \tan x \tan^2 x + 4\sec^2 x \cdot 2\tan x \sec^2 x + 8\sec^3 x \sec x \tan x$

$= 8\sec^2 x \tan^3 x + 16 \sec^4 x \tan^2 x$

$f(0) = 0 \quad f'(0) = 1 \quad f''(0) = 0 \quad f'''(0) = 2$

$$\boxed{T_3(x) = x + \frac{1}{3}x^3}$$

b. $|R_3(x)| = \dfrac{f^{(4)}(z)}{4!}(x)^4$

$\sec(\pi/6) = 2/\sqrt{3}$

$\tan \pi/6 = 1/\sqrt{3}$

$f^{(4)}(0) = 0$ increasing to $f^{(4)}(\pi/6) = \dfrac{8 \cdot \frac{4}{3} \cdot \frac{1}{3\sqrt{3}} + 16 \cdot \frac{16}{9} \cdot \frac{1}{3}}{24}$

$= \dfrac{2.055 + 1.185}{24}$

$= .135$

$.134 \cdot \pi/6 \approx .07$

13. $f(x) = e^{x^2}$ $f'(x) = 2xe^{x^2}$ $f''(x) = 2e^{x^2} + 4x^2e^{x^2}$

$f'''(x) = 4xe^{x^2} + 8xe^{x^2} + 8x^3e^{x^2}$

$= 12xe^{x^2} + 8x^3e^{x^2} = (12x + 8x^3)e^{x^2}$

$f^{(4)}(x) = (12 + 24x^2)e^{x^2} + 2x(12x + 8x^3)e^{x^2}$

$= (12 + 48x^2 + 8x^3)e^{x^2}$

$f(0) = 1$ $f'(0) = 0$ $f''(0) = 2$ $f'''(0) = 0$

$T_3(x) = 1 + \frac{2x^2}{2} = 1 + x^2$

$|R_4(x)| = \frac{f^{(4)}(x)}{4!}x^4 \le \frac{(12 + .1^2 \cdot 48 + 8 \cdot .1^3)e^{.01}}{4!} \cdot (.01)$

$= .50004\, e^{.01} \cdot .01$

$= .005$

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21. $\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} - \dots$

Thus by A.S.E.T. the error using $x - \frac{x^3}{6}$ is $< \left|\frac{x^5}{120}\right|$.

Need $\left|\frac{x^5}{120}\right| < .01$

$|x^5| < 1.2$

$|x| < 1.037$

$-1.037 < x < 1.037$

22. As in # 21 Need

$\left|\frac{x^4}{720}\right| < .005$

$|x^4| < 3.6$

$|x| < 1.238$ $-1.238 < x < 1.238$