Name: SOLUTIONS

Math 2950- Midterm Exam #3 - November 23, 2004

1. **(15 points)** Calculate the work done by $\mathbf{F}(x,y,z) = (y-x^2, z-y^2, x-z^2)$ over the curve $\mathbf{r}(t) = (t, t^2, t^3)$ from $(0,0,0)$ to $(1,1,1)$.

$$W = \int_0^1 F(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt = \int_0^1 (0, t^3 - t^4, t - t^6) \cdot (1, 2t, 3t^2)\,dt$$

$$= \int_0^1 2t^4 - 2t^5 + 3t^3 - 3t^8\,dt$$

$$= \frac{2}{5}t^5 - \frac{1}{3}t^6 + \frac{3}{4}t^4 - \frac{1}{3}t^9 \Big|_0^1$$

$$= \frac{2}{5} - \frac{1}{3} + \frac{3}{4} - \frac{1}{3} = 29/60$$

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2. **(15 points)** Evaluate the line integral: $\int_C x\,dx + y\,dy$ where C consists of the top half of the circle $x^2 + y^2 = 1$ from $(1,0)$ to $(-1,0)$.

$$\vec{r}(t) = (\cos t, \sin t) \quad 0 \le t \le \pi$$

$$x'(t) = -\sin t \quad y'(t) = \cos t$$

$$\int_0^\pi \cos t(-\sin t)\,dt + \sin t \cos t\,dt$$

$$= \frac{1}{2}\cos^2 t + \frac{1}{2}\sin^2 t \Big|_0^\pi = \frac{1}{2}\Big|_0^\pi = 0$$

3. **(15 points)** Calculate $\iiint_E z\,dV$ where E is the first octant region between the spheres $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 9$.

In spherical: $0 \le \theta \le \pi/2 \quad 0 \le \phi \le \pi/2 \quad 1 \le \rho \le 3$

$$\int_1^3 \int_0^{\pi/2} \int_0^{\pi/2} \rho\cos\phi \cdot \rho^2 \sin\phi \, d\phi\, d\theta\, d\rho = \int_1^3 \rho^3 d\rho \int_0^{\pi/2} \sin\phi\cos\phi\, d\phi \int_0^{\pi/2} 1\, d\theta$$

$$= \frac{\rho^4}{4}\Big|_1^3 \; \frac{\sin^2\phi}{2}\Big|_0^{\pi/2} \cdot \theta\Big|_0^{\pi/2}$$

$$= \left(\frac{80}{4}\right)\left(\frac{1}{2}\right)\cdot\frac{\pi}{2} = \frac{80\pi}{16}$$

$$= 5\pi$$

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4. **(20 points)** Use the change of coordinates:

$$u = x - y, \; v = 2x + y$$

to evaluate the integral

$$\iint_R (2x^2 - xy - y^2)\, dx\, dy$$

over the parallelogram R bounded by the lines $\underset{v=4}{\underline{y + 2x = 4}}$, $\underset{v=7}{\underline{y + 2x = 7}}$, $\underset{u=-2}{\underline{y - x = 2}}$ and $\underset{u=-1}{\underline{y - x = 1}}$.
Hint: $2x^2 - xy - y^2$ factors.

$u = x - y \Rightarrow 3x = u + v \qquad x = \frac{1}{3}u + \frac{1}{3}v$

$v = 2x + y \Rightarrow v - 2u = 3y \qquad y = -\frac{2}{3}u + \frac{1}{3}v$

$$J = \begin{vmatrix} 1/3 & 1/3 \\ -2/3 & 1/3 \end{vmatrix} = 1/9 + 2/9 = 1/3$$

$2x^2 - xy - y^2 = uv$

$$\int_{-2}^{-1}\int_4^7 uv\cdot\frac{1}{3}\, dv\, du = \frac{1}{3}\cdot\int_{-2}^{-1} u\,du \int_4^7 v\,dv$$

$$= \frac{1}{3}\cdot\frac{u^2}{2}\Big|_{-2}^{-1}\cdot\frac{v^2}{2}\Big|_4^7$$

$$= \frac{1}{3}\left(\frac{-3}{2}\right)\left(\frac{33}{2}\right) = \frac{-33}{4}$$

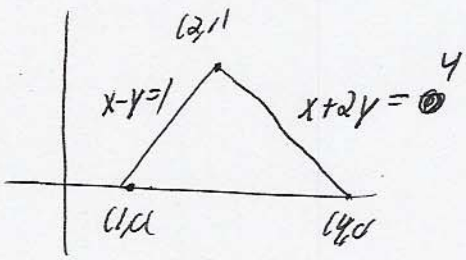
5. **(15 points)** Set S be the part of the sphere $x^2+y^2+z^2=9$ which lies above the xy plane and *inside* the cylinder $x^2+y^2=4$. Set up, but do not evaluate, a double integral in polar coordinates which gives the surface area of S.

$$z=\sqrt{9-x^2-y^2} \qquad \frac{\partial z}{\partial x}=\frac{-x}{\sqrt{9-x^2-y^2}} \qquad \frac{\partial z}{\partial y}=\frac{-y}{\sqrt{9-x^2-y^2}}$$

$$\sqrt{1+\frac{\partial z}{\partial x}^2+\frac{\partial z}{\partial y}^2}=\sqrt{1+\frac{x^2+y^2}{9-x^2-y^2}}=\sqrt{\frac{9}{9-x^2-y^2}}$$

$$S.A.=\int_0^2\int_0^{2\pi}\sqrt{\frac{9}{9-r^2}}\;r\,d\theta\,dr$$

6. **(20 points)** Find the volume under the surface $z=xy$ and above the triangle in the xy plane with vertices $(1,0)$, $(2,1)$ and $(4,0)$.



So $1+y \le x \le 4-2y$

$0 \le y \le 1$

$$V=\int_0^1\int_{1+y}^{4-2y} xy\,dx\,dy=\int_0^1 \frac{y}{2}\cdot x^2\Big|_{1+y}^{4-2y}\,dy$$

$$=\frac{1}{2}\int_0^1 y\left((4-2y)^2-(1+y)^2\right)dy$$

$$=\frac{1}{2}\int_0^1 y\left(4y^2-8y+16-y^2-2y-1\right)dy$$

$$=\frac{1}{2}\int_0^1 4y^3-8y^2+16y-y^3-2y^2-y\,dy$$

$$=\frac{1}{2}\int_0^1 3y^3-10y^2+15y\,dy$$

$$=\frac{1}{2}\left(\frac{3}{4}y^4-\frac{10}{3}y^3+\frac{15}{2}y^2\right)\Big|_0^1$$

$$=\frac{1}{2}\left(\frac{3}{4}-\frac{10}{3}+\frac{15}{2}\right)=\frac{1}{2}\left(\frac{9-40+90}{12}\right)$$

$$=\frac{59}{24}$$