

Math 3320 Homework #12, Due Tuesday March 1

1. Let $H \leq G$ and let $g \in G$. Consider the set:

$$gHg^{-1} = \{ghg^{-1} \mid h \in H\}.$$

- Show that gHg^{-1} has the same cardinality as H .
- Prove that gHg^{-1} is a subgroup of G .
- Let $H = \{e, (12), (34), (12)(34)\} \leq S_4$ and let $g = (123)$. Calculate the subgroup gHg^{-1} .

2. List all subgroups of D_4 and determine which are normal.

3. Let $H \leq G$. We define a subset of G called the *normalizer* of H in G as:

$$N_G(H) = \{g \in G \mid gHg^{-1} = H\}$$

- Prove that $N_G(H) \leq G$.
 - If $H \triangleleft G$, what is $N_G(H)$?
4. Let $G = GL_n(R)$ be the group of invertible $n \times n$ matrices with real entries. Recall these are exactly the matrices with nonzero determinant. Let $S = SL_n(R)$ be the subgroup of matrices with determinant equal to 1.

- Show that S is a normal subgroup of G .
- Give a criterion for two matrices in G to be in the same coset of S .

5. Recall from the exam that elements x and y are said to be *conjugate* if $x = gyg^{-1}$ for some $g \in G$. Prove that conjugate elements have the same order. Hint: What happens when you raise gyg^{-1} to a power?

6. Use Lagrange's theorem to prove that if a group G has order n and $g \in G$ then $g^n = e$. Note we are not saying every element of the group has order n , just that every element raised to the n th power is the identity. The order is the *minimal* power which gives e .

7. Prove that the intersection of two normal subgroups is normal.