Name:

Math 3320 Midterm Exam #1 - February 10, 2005

1. **(15 points)** Complete the following definitions:

   a. Let G and G' be two groups. A *homomorphism* from G to G' is...

   b. Let G and G' be two groups. An *isomorphism* from G to G' is...

   c. Let G be a group with $g \in G$. The *order* of g is...

   d. The symmetric group S_n is...

   e. Let G be a group with $g \in G$. The *cyclic subgroup generated by* g is...

2. **(16 points)** True or false:

____ a. Every group of order four is abelian.

____ b. Every group of order four is cyclic.

____ c. The odd permutations in S_n form a subgroup.

____ d. The intersection of two subgroups of a group is always a subgroup.

____ e. The union of two subgroups of a group is always a subgroup.

____ f. Let $g, h, x \in G$. If $gx = xh$ then $g = h$.

____ g. Subtraction is associative.

____ h. A group has only one identity element.

3. **(16 points)** Determine whether the given binary structure is a group. If not give a group axiom that fails:

   a. $U = \{z \in \mathbf{C} \mid |z| = 1\}$ under multiplication

   b. $U = \{z \in \mathbf{C} \mid |z| \geq 1\}$ under multiplication.

   c. $U_n = \{z \in \mathbf{C} \mid z^n = 1\}$ under multiplication.

   d. $\{3^n \mid n \in \mathbf{Z}\}$ under addition.

   e. $\{1, 3, 4, 5, 7\}$ under multiplication mod 8.

f. SL_n= $\{n \times n$ matrices with determinant $1\}$ under matrix multiplication.

g. $H = \{\sigma \in S_7 \mid \sigma(1) = 1.$

h. $H = \{\sigma \in S_7 \mid \sigma(1) = 2.$

4. **(20 points)** Let G be a group and let $x, y \in G$. We say x is *conjugate to* y if there exists some $g \in G$ such that: $gxg^{-1} = y$. If x is conjugate to y then write $x \sim y$.

a. Show that the only element conjugate to e is e.

b. Show that if G is abelian and $x \in G$ then the only element conjugate to x is x.

c. Give the properties of $\sim$ needed for it to be an equivalence relation.

d. Prove that $\sim$ is an equivalence relation on an arbitrary group G. (The equivalence classes are called *conjugacy classes.*)

5. **(12 points)**

a. What are the generators for the cyclic group Z_{18} under $+_{18}$?

b. Give the complete subgroup lattice of Z_{18}.

6. **(15 points)** Let $\phi : G \to G'$ be a group homomorphism. Define:

$$K = \{g \in G \mid \phi(g) = e\}.$$

i.e. K is all the elements that map to the identity under ϕ. K is called the *kernel* of the homomorphism.

a. Prove that K is a subgroup of G.

b. Now suppose $G = \mathbf{C}^*$, the nonzero complex numbers under multiplication, and suppose $G' = \mathbf{R}^+$, the positive real numbers under multiplication. Recall the norm map:

$$\phi : \mathbf{C}^* \to \mathbf{R}^+$$

given by $\phi(z) = |z|$. Check that ϕ is a homomorphism. Determine the kernel. Is it a subgroup of $\mathbf{C}^*$?

7. **(10 points)** The multiplication table is given below for the *quaternion group* Q.

	1	-1	i	j	k	-i	-j	-k
1	1	-1	i	j	k	-i	-j	-k
-1	-1	1	-i	-j	-k	i	j	k
i	i	-i	-1	k	-j	1	-k	j
j	j	-j	-k	-1	i	k	1	-i
k	k	-k	j	-i	-1	-j	i	1
-i	-i	i	1	-k	j	-1	k	-j
-j	-j	j	k	1	-i	-k	-1	i
-k	-k	k	-j	i	1	j	-i	-1

a. Is Q abelian?

b. Find the order of the elements -1, i, and -k.

c. Find the inverse of i.

d. Find the subgroup generated by k.

e. (Bonus) Give a reason why this group is not isomorphic to the dihedral group D_4 (symmetries of a square).