Math 3320 Midterm Exam #2 - March 17, 2005

1. **(8 points)** Complete the following definitions:

a. The *index* of a subgroup is...

b. Let $\phi : G \to G'$ be a group homomorphism. The *kernel* of ϕ is...

c. A subgroup $H \leq G$ is *normal* if ...

d. A *simple group* is...

2. **(14 points)** True or false:

____ a. Every subgroup of an abelian group is normal.

____ b. If G is abelian and $H \trianglelefteq G$ then the factor group G/H is abelian.

____ c. If $H \trianglelefteq G$ and if G/H is abelian then G must be abelian.

____ d. The intersection of two normal subgroups of a group is always a normal subgroup.

____ e. If $H \trianglelefteq G$ then $gh = hg$ for all $g \in G$ and for all $h \in H$.

____ f. The direct product of two cyclic groups is always cyclic.

____ g. Every normal subgroup is the kernel of some homomorphism.

3. **(12 points)**

a. Let H and K be groups. Explain how to construct the direct product group $H \times K$. That is, explain what the elements of the direct product are and what the operation is.

b. If H and K are finite groups, how many elements does the group $H \times K$ have?

c. Find the order of all elements in the group $Z_2 \times Z_3$. Is it cyclic?

4. **(12 points)** Let $\sigma = (123)(45)$ and $\tau = (1342)$ be elements of S_5. Write the following in disjoint cycle notation:

$$\sigma\tau, \ \ \tau\sigma^2, \ \ \tau\sigma\tau^{-1}, \ \ \sigma^{38}\tau^{101}$$

5. **(15 points)** Let H and N be subgroups of G. Recall the *set*:

$$HN = \{hn \mid h \in H, n \in N\}.$$

Hint: Both parts of this problem may benefit from variations of the trick of inserting gg^{-1}.
a. Prove that if N is normal in G then HN is a subgroup of G.
b. Now assume *both* N and H are normal and prove that HN is normal.

Choose either 6 or 6'. Clearly indicate your choice!

6. **(12 points)** Let $\phi : G \to G'$ and $\gamma : G' \to G''$ be group homomorphisms. Carefully show the composite map $\gamma\phi : G \to G''$ is also a homomorphism.

6'. Let $\phi : G \to G'$ be a group homomorphism. Prove that ϕ is 1-1 if and only if $\ker \phi = \{e\}$.

7. **(15 points)** The multiplication table is given below for the *quaternion group* Q.

	1	-1	i	j	k	-i	-j	-k
1	1	-1	i	j	k	-i	-j	-k
-1	-1	1	-i	-j	-k	i	j	k
i	i	-i	-1	k	-j	1	-k	j
j	j	-j	-k	-1	i	k	1	-i
k	k	-k	j	-i	-1	-j	i	1
-i	-i	i	1	-k	j	-1	k	-j
-j	-j	j	k	1	-i	-k	-1	i
-k	-k	k	-j	i	1	j	-i	-1

a. Notice that $Z = \{\pm 1\}$ is a subgroup. Write down the four left cosets and four right cosets to verify that $Z \trianglelefteq Q$.

b. Construct and clearly label the multiplication table for the quotient group Q/Z.

c. What familiar group is Q/Z isomorphic to?

8. **(12 points)** Let $x, y \in G$. Define a new element of G, the *commutator* of x and y, denoted by $[x, y] = x^{-1}y^{-1}xy$.

a. Prove that a group G is abelian if and only if all commutators are equal to the identity, i.e. $x^{-1}y^{-1}xy = e$ for all $x, y \in G$.

b. Now suppose $H \trianglelefteq G$. Prove that G/H is abelian if and only if $[x, y] \in H$ for all $x, y \in G$. Hint: What does it mean for two elements of G/H to commute?