Math 3820- Final Exam - May 2, 2005

1. **(20 points)** Solve the differential equation below by means of a power series around the point x_0. Find the recurrence relation for the coefficients and find the first four terms (unless the series terminates sooner) in each of two linearly independent solutions. What can you say about the radius of convergence of your solutions?

$$(4 - x^2)y'' + 2y = 0,\ x_0 = 0.$$

2. (**10 points)** Determine a lower bound for the radius of convergence of series solutions about each of the three points x_0 for the differential equation:

$$(x^2 - x - 6)y'' + xy' - 5y = 0;\ x_0 = 0,\ x_0 = 1,\ x_0 = -5.$$

3. (**10 points)** Determine a general solution that is valid in any interval not including the singular point:

$$x^2y'' - 3xy' + 4y = 0.$$

4. (**20 points)** Solve the initial value problem completely (i.e. get an equation for $y(x)$) and determine the values of x for which the solution is defined.

$$y' = \frac{xy^3}{\sqrt{1 + x^2}},\ y(0) = 1$$

5. **(20 points)** Solve the initial value problem:

$$y' + (\frac{2}{t})y = \frac{\cos(t)}{t^2},\ y(\pi) = 1,\ t > 0.$$

6. **(20 points)** Use the method of undetermined coefficients to find a general solution to:

$$y'' + 4y = t + \sin(2t)$$

7. **(20 points)** For each function below find a constant coefficient second order linear differential equation having it as a solution. For example given $\cos(t)$ then you might give $y'' + y = 0$ as your equation. (Hint: what are the roots of the characteristic equation in each case?)

a. $\cos(3t) + \sin(3t)$
b. te^{4t}
c. $e^{-2t} + 3e^{t}$.
d. $e^{t}\cos(t) + 2e^{t}\sin(t)$

8. **(15 points)** Below is an integro-differential equation. Determine $\mathcal{L}\{y\}$ where y is a solution. You do not need to determine y.

$$y' + y = \int_0^t y(v)(t-v)^2 dv + t \quad y(0) = 1.$$

9. **(15 points)** Use the definition of the Laplace transform to prove line 14 on your Laplace transform table, i.e. prove that if $\mathcal{L}\{f(t)\} = F(s)$ then $\mathcal{L}\{e^{ct}f(t)\} = F(s-c)$

10. **(20 points)** Find the solution of the initial value problem:

$$y'' + y = \begin{cases} t/2 & 0 \le t < 6 \\ 3 & 6 \le t < \infty; \end{cases} \quad y(0) = 0, \; y'(0) = 1$$

11. **(15 points)**
Find the inverse Laplace transform $\mathcal{L}^{-1}\{F(s)\}$ of

$$F(s) = 6 + \frac{s}{s^2 + 4s + 13}$$

12. **(15 points)** Explain the difference between linear and nonlinear equations as relates to the existence and uniqueness of solutions to the initial value problem:

$$y' = f(t, y), \quad y(t_0) = y_0$$