Math 4330- Final Exam - May 2, 2006

Directions: The exam is worth 200 points. Put all work to be graded in the blue book. Also let $\mathbb{Q}, \mathbb{R}, \mathbb{C}$ denote the field of rational numbers, real numbers and complex numbers respectively.

1. **(12 points)** Define the following terms:

   a. Let F be a field and $f(x) \in F[x]$. The *Galois group of f(x)* is...

   b. A group G is *solvable* if...

   c. Let E be a field. A *field automorphism of E* is...

   d. A group G is *simple* if...

   e. An *integral domain* is...

   f. A binary operation $*$ on a set S is *associative* if...

2. **(22 points)** True or false:

____ a. The field extension $\mathbb{Q}(\sqrt[3]{5}) : \mathbb{Q}$ is normal.

____ b. The field extension $\mathbb{C} : \mathbb{R}$ is normal.

____ c. Suppose $p(x) \in F[x]$ has degree n. Then the Galois group of $p(x)$ has at most n elements.

____ d. Let $p(x) \in F[x]$. There is always some extension field E of F in which $p(x)$ has a root.

____ e. Using only a compass and straightedge it is possible to construct a $112.5°$ angle.

____ f. All groups of order 11 are isomorphic.

____ g. The smallest nonabelian group has 6 elements.

____ h. The nonzero elements in a field form an abelian group under multiplication.

____ i. The Klein 4-group is cyclic.

____ j. Suppose $H \trianglelefteq G$ and both H and G/H are abelian. Then G must be abelian.

____ k. $\mathbb{Q}(\sqrt{2},\sqrt{3}) : \mathbb{Q}$ is a simple extension.

3. **(12 points)**
   a. State Lagrange's Theorem.
   b. State Cauchy's Theorem.

4. **(21 points)** Give an example of:

a. A noncommutative ring without identity.

b. A commutative ring which is not an integral domain.

c. A nonabelian simple group.

d. A zero divisor in the ring $M_2(\mathbb{R})$.

e. A polynomial in $\mathbb{Z}[x]$ which has degree 5, leading coefficient 15, and is irreducible by the Eisenstein Criterion.

f. A primitive 4th root of unity in $\mathbb{C}$.

g. A normal subgroup of S_4 which is not $\{e\}$ or S_4.

5. **(24 points)** For each α and field F below, do the following:

i. Determine the minimal polynomial of α over F.

ii. Determine the splitting field E of the minimal polynomial.

iii. Give the degree $[E : F]$ of the extension and a linear basis for E over F.

a. $\alpha = \sqrt{7}$, $F = \mathbb{Q}$

b. $\alpha = i$, $F = \mathbb{R}$

c. $\alpha = \sqrt[3]{2}$, $F = \mathbb{Q}$

d. $\alpha = \pi^2$, $F = \mathbb{Q}(\pi^4)$.

6. **(12 points)** For each E below, give a linear basis of E over F. Then determine the Galois group $G(E/F)$.

a. $F = \mathbb{Q}$, E is the splitting field of $(x^2 - 2)(x^2 - 3)$.

b. $F = \mathbb{Q}$, $E = \mathbb{Q}(\sqrt[3]{7})$.

7. **(15 points)** Let $H \trianglelefteq G$ and let $m = [G : H]$. Prove $g^m \in H$ for all $g \in G$.

8. **(10 points)** Recall that a real number α is constructible if one can construct a line segment of length $|\alpha|$ in finite many allowable steps using a compass and straightedge and starting with a line segment of length 1. Which of the following are constructible?

$$\sqrt[3]{7}, \quad \frac{113}{25}, \quad \pi^2, \quad e, \quad \sqrt{1-\sqrt[4]{3}}, \quad \sin(15°), \quad \sqrt{\frac{2}{3}}, \quad \cos(10°)$$

9. **(16 points)** Let $\mathbb{Z}_2 = \{0, 1\}$ be a field of two elements and let Let $p(x) = x^2 + x + 1 \in \mathbb{Z}_2[x]$.

a. Prove $p(x)$ is irreducible over $\mathbb{Z}_2$.

b. Construct an extension field $\mathbb{Z}_2 \subset E$ in which $p(x)$ has a root. Give a complete addition and multiplication table for the field E.

c. Does $p(x)$ split over E?

d. Find an irreducible degree two polynomial in $E[x]$.

10. **(15 points)** Let $\phi : G \to H$ be a group homomorphism with kernel K. Let $g \in G$. Prove the *equality of sets* below:

$$\phi^{-1}[\phi(g)] = gK.$$

11. **(10 points)**

a. Let G be a finite group and $g \in G$. Define the *order* of g.

b. Now let G be the symmetric group S_{15}. Write down, in disjoint cycle notation, an element $\sigma \in S_{15}$ which has the largest possible order of any element in S_{15}. (There are many choices for σ.)

12. **(15 points)** Let $F \subset E$ be a normal extension with Galois group $G = G(E/F)$. The fundamental theorem of Galois theory gives a correspondence between subgroups of G and intermediate fields $F \subseteq E_1 \subseteq E$.

a. Given a subgroup $H \leq G$, explain how one obtains a corresponding intermediate field $F \subseteq E_1 \subseteq E$

b. Given an intermediate field E_1 with $F \subseteq E_1 \subseteq E$, explain how one obtains a corresponding subgroup of G.

13. **(16 points)** Let $\phi : G \to H$ be a group homomorphism which is onto and let K be the kernel.
a. Prove $K \trianglelefteq G$.
b. What can you say about the group G/K?
c. If $g \in G$ has order 6 what can you say about the order of $\phi(g)$?