2. We want surjections from a 10 element set to a 5 elt. set. In class we used I/E to get a formula for this:

$$\sum_{i=0}^{5} (-1)^i \binom{5}{i} (5-i)^{10}$$

$$= 5^{10} - 5 \cdot 4^{10} + 10 \cdot 3^{10} - 10 \cdot 2^{10} + 5 \cdot 1^{10} - 0$$

$$= 9765625 - 5242880 + 590490 - 20480 + 5$$

$$= 5111192$$

3. Let A, S, B be cars w/ auto trans, power steering, power brakes

<u>Given</u> $|A| = 9$ $|S| = 12$ $|B| = 8$

$|A \cap S| = 7$ $|A \cap B| = 4$ $|S \cap B| = 5$

$|A \cap S \cap B| = 3$

- automatic transition only:

want $|A| - |A \cap B| - |A \cap S| + |A \cap B \cap S|$

$= 9 - 4 - 7 + 3 = 1$

no features at all $= 18 - |A \cup S \cup B|$

$|A \cup S \cup B| = 9 + 12 + 8 - 7 - 4 - 5 + 3 = 16$

so 2 cars

brake but NO auto trans =

$|B| - |B \cap A| = 8 - 4 = 4$

4. Let $d(n)$ = # of derangements of n.

a. $Prob = \dfrac{d(n)}{n!} \cdot \dfrac{d(n)}{n!}$ so $\lim_{n \to \infty} P = 1/e^2$

b. Let B_i, $i=1,\dots,n$, be the # of ways to pass out the papers so that student i gets his quiz and his HW. There are $(n!)^2$ ways to pass out the papers so we want to know:

$$P = \frac{n!^2 - |B_1 \cup \cdots \cup B_n|}{(n!)^2}$$

$|B_1| = (n-1)!^2 \quad |B_1 \cap B_2 \cap \cdots \cap B_k| = (n-k)!^2$ so

$$P = \frac{n!^2 - \left(\sum_{i=1}^{n} (-1)^{i+1} \binom{n}{i} (n-i)!^2\right)}{n!^2} \quad \text{by inc/exc}$$

But $\binom{n}{i}(n-i)!^2 = \dfrac{n!}{i!(n-i)!}(n-i)!^2 = \dfrac{n!(n-i)!}{i!}$

so

$$P = \frac{n!^2 - \sum_{i=1}^{n} (-1)^{i+1} \frac{n!(n-i)!}{i!}}{(n!)^2} = \frac{n! - \sum_{i=1}^{n} (-1)^{i+1} \frac{(n-i)!}{i!}}{n!}$$

$$= \frac{\sum_{i=0}^{n} (-1)^{i} \frac{(n-i)!}{i!}}{n!} \quad \text{since } (-1)^0 \frac{(n-0)!}{0!} = n!$$

So we get our answer

$$\boxed{P = \sum_{i=0}^{n} (-1)^i \frac{(n-i)!}{n!\, i!}}$$

As $n \to \infty$ this converges to 1, (Done on MAPLE)

- so as n gets large, with probability approaching 1, no student gets both their papers back.

5.

Prove: $d(n) = (n-1)[d(n-1) + d(n-2)]$

Recall a derangement is a permutation of $\{1,2,..,n\}$ π such that $\pi(i) \neq i$ $\forall i$. Thus there are $n-1$ choices for $\pi(1)$ so $d(n) = (n-1) \cdot [\#$ derangements with $\pi(1)=2]$.
So we need to prove:

Lemma The # of derangements so that $\pi(1)=2$ is $d(n-1)+d(n-2)$.

Proof If $\pi(2)=1$ then we need a derangement of 3, 4, 5, .., n. Thus
$* \ d(n-2) = \#$ of der. w/ $\pi(1)=2, \pi(2)=1$.

Suppose $\pi(2) \neq 1$. We have:

$$\begin{array}{lcccccc} i: & 1 & 2 & 3 & 4 & \cdots & n \\ \pi(i): & 2 & \underline{a} & \cdots & & 1 & \cdots \end{array}$$

where $a \neq 1$. But the # of ways to fill in the bottom row above with $1, 3, 4, .., n$ so $a \neq 1$ and it is still a derangement is just $d(n-1)$. Thus we get a total of $d(n-1)+d(n-2)$, proving the lemma.

Claim $d(n) = n\,d(n-1) + (-1)^n$

Proof By induction, $d(2)=1$ & $d(1)=0$ so works for $n=2$
Suppose it holds for all $d(k)$ $k<n$. Then

$$d(n) = n\,d(n-1) - d(n-1) + n\,d(n-2) - d(n-2) \quad \text{by part 4,}$$

$$= n\,d(n-1) - \left((n-1)d(n-2) + (-1)^{n-1}\right) + (n-1)d(n-2)$$

by inductive hypothesis.

$$= n\,d(n-1) - (-1)^{n-1} = n\,d(n-1) + (-1)^n. \quad //$$