Math 6980 Homework # 1, Assigned 1/10/06, Due 1/17/06

1. Recall (or trust me) that if k is a finite field, then k is determined up to isomorphism by its order $|k| = q$ where q must equal p^a for some prime p. It is typical then to write $GL(n, q)$ rather than $GL(n, k)$ for the *finite* group of invertible $n \times n$ matrices with entries from k.

a. Let k be a field of q elements and let V be an n-dimensional vector space over k. Show that V contains exactly q^n vectors.

b. Recall that a square matrix is invertible if and only if its columns are linearly independent. Thus for a matrix in $GL(n, q)$ the first column can be any of the $q^n - 1$ nonzero vectors in k^n and each succeeding column cannot be in the subspace spanned by the previous columns. Using this as a hint prove that:

$$|GL(n,q)| = \prod_{k=1}^{n} (q^n - q^{k-1}) = q^{\frac{n(n-1)}{2}} (q^n - 1) \cdots (q - 1).$$

c. Let $k = \{0, 1\}$ be the field of two elements, which is just the integers mod 2. (So $1 + 1 = 0$). Write down the matrices in $GL(2, 2)$.

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3. Let V and W be vector spaces with bases $\{v_1, v_2, v_3\}$ and $\{w_1, w_2\}$ respectively. Let $T_1 : V \to V$ be the linear transformation given by the matrix $\begin{pmatrix} 0 & 2 & -1 \\ 1 & 0 & 3 \\ 2 & -1 & 1 \end{pmatrix}$ and let $T_2 : W \to W$ be the linear transformation given by the matrix $\begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix}$ with respect to these bases.

a. Write down an ordered basis for the vector space $V \otimes W$.

b. Write the matrix for the linear map

$$T_1 \otimes T_2 : V \otimes W \to V \otimes W$$

in terms of this basis

4. Let A and B be $n \times n$ matrices. Prove that trace(AB) = trace(BA). Then prove that similar matrices have the same trace.

5. Let V be a finite-dimensional vector space over the field F and let W be a subspace of V. If $f \in W^*$ is a linear functional on W, prove that there is a linear functional $g \in V^*$ such that $g(w) = f(w) \forall w \in W$.