# Math 6980 Homework #1 solutions

1a. Let $V$ have basis $\{v_1, v_2, \ldots, v_n\}$. Then every $v \in V$ is a unique linear combination of the $\{v_i\}$ so:

$$V = \{c_1 v_1 + \cdots + c_n v_n \mid c_i \in k\}.$$

Since $k$ has $q$ elements there are $q^n$ possible $n$-tuples of $c_i's$ so $|V| = q^n$.

b. Let $A \in GL(n, q)$, which means the columns of $A$ are linearly independent. The first column can be any of the $q^n - 1$ nonzero vectors in $k^n$. The second column must not be in the span of the first column, which contains $q$ vectors. Thus there are $q^n - q$ choices. The third column cannot be among the $q^2$ vectors spanned by the first two columns, so there are $q^n - q^2$ choices, etc.. Thus we get:

$$|GL(n, q)| = \prod_{i=0}^{n-1} (q^n - q^i).$$

c. By (b) we know $GL(2, 2)$ has 6 elements. They are:

$$\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\}$$

Notice this group is isomorphic to $\Sigma_3$.

B9. First observe that if $A^\bullet$ and $B^\bullet$ are graded algebras then so is $A \otimes B$ with

$$(A \otimes B)_k = \oplus_{j=0}^{k} A_j \otimes B_{k-j}$$

In particular the first graded piece of $\mathrm{Sym}^\bullet V \otimes \mathrm{Sym}^\bullet W$ is $(V \otimes \mathbf{C}) \oplus (\mathbf{C} \otimes W)$. Define a linear map from $V \oplus W$ to $(V \otimes \mathbf{C}) \oplus (\mathbf{C} \otimes W)$ by extending the obvious maps from $V$ and $W$ linearly. By the universal property this determines a homomorphism of graded algebras:

$$\Phi : \mathrm{Sym}^\bullet(V \oplus W) \to \mathrm{Sym}^\bullet V \otimes \mathrm{Sym}^\bullet W.$$

This map can be seen to be an isomorphism since it takes the standard basis on the left to the standard basis on the right. The component in the $n$th homogeneous term gives the isomorphism from $B2$.

The second part is similar.

3a. $\{v_1 \otimes w_1, v_1 \otimes w_2, v_2 \otimes w_1, v_2 \otimes w_2, v_3 \otimes w_1, v_3 \otimes w_2\}$

b. $\begin{pmatrix} 0 & 0 & 2 & -2 & -1 & 1 \\ 0 & 0 & 4 & -2 & -2 & 1 \\ 1 & -1 & 0 & 0 & 3 & -3 \\ 2 & -1 & 0 & 0 & 6 & -3 \\ 2 & -2 & -1 & 1 & 1 & -1 \\ 4 & -2 & -2 & 1 & 2 & -1 \end{pmatrix}$

4. The trace statement is an easy calculation. Now $QAQ^{-1} = (QA)Q^{-1}$ so trace$(QAQ^{-1})$ = trace$(Q^{-1}QA)$=trace$(A)$, so similar matrices have the same trace.

5. Pick a basis of $W$ and extend to a basis of $V$. Define $g : V \to k$ so that $g$ agrees with $f$ on the basis of $W$. On the remaining basis vectors $g$ can be arbitrary.

**Exercise 1.1**

Let $f \in V^*$. The pairing in question is given by $< f, v >= f(v)$. To get:

$$< \rho^*(g)(v^*), \rho(g)(v) >=< v, v^* >$$

it is enough to check on a basis so let $\{v_1, v_2, \ldots, v_n\}$ be a basis of $V$ with dual basis $\{\tilde{v}_1, \tilde{v}_2, \ldots, \tilde{v}_n\}$ Let $\rho(g)$ have matrix $A$ and $\rho^*(g)$ have matrix $B$ in terms of these basis. Since $< \tilde{v}_j, v_i >= \delta_{ij}$ we need:

$$< \rho^*(g)(\tilde{v}_j), \rho(g)(v_i) >= \delta_{ij}.$$

Expanding out we get:

$$< \sum_s B_{sj}\tilde{v}_s, \sum_k A_{ki}v_k >= \delta_{ij}$$

which gives:

$$\sum_{s=1}^{n} B_{sj}A_{si} = \delta_{ij},$$

i.e. ${}^{\tau}BA = I$, so $B =^{\tau} A^{-1}$ as defined.

**Exercise 1.3** The matrix $\rho(g)$ acts on the (one-dimensional) top exterior power $\Lambda^n(V)$ by the determinant. Hence the assumption is equivalent to the statement that $\Lambda^n(V) \cong \mathbf{C}$ as $G$-modules. Now see the remark on the bottom of page 28 about perfect pairings. The wedge product gives a bilinear map $\Lambda^k(V) \times \Lambda^{n-k}(V) \to \Lambda^n(V) \cong \mathbf{C}$. For any basis vector $v_{j_1} \wedge v_{j_2} \wedge \cdots \wedge v_{j_k}$ of $\Lambda^k(V)$ there is (exactly one) basis vector of $x \in \Lambda^{n-k}(V)$ such that $v_{j_1} \wedge v_{j_2} \wedge \cdots \wedge v_{j_k} \wedge x$ is nonzero, so this is a perfect pairing. ($x$ is just the wedge of the remaining $n-k$ basis vectors of $V$.) Thus it induces an isomorphism:

$$\Lambda^k(V) \cong \Lambda^{n-k}(V)^* \cong \Lambda^{n-k}(V^*).$$

**Exercise 1.4** a. Let $e_x$ be as described. For the descriptions to obvious bijection $x \to e_x$ needs to be a $G$-map, i.e. we need $ge_x = e_{xg}$. We just check on $y \in G$:

$$(ge_x)(y) = e_x(g^{-1}y) = \delta_x^{g^{-1}y} = \delta_{gx}^{y}$$

so $ge_x = e_{gx}$.

b. Easy to check this using $(xy)^{-1} = y^{-1}x^{-1}$.