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10/10/06

Direct Products

Def $G_1, G_2, \ldots, G_n$ groups $G_1 \times G_2 \times \cdots \times G_n$ is new group
1. Elts $\{(g_1, \ldots, g_n) \mid g_i \in G_i\}$, coord op

Remarks
1. $|G_1 \times \ldots \times G_n| = |G_1||G_2| \cdots |G_n|$
2. ∞ly many : direct product vs direct sum aka restricted direct product
example $C_2 \times C_3 \times C_5 \times C_7 \times C_{11} \times \ldots$

Elementary Properties
1. $G = \prod_{i=1}^{n} G_i$ contains an $\cong$ copy of G_i

$$G_i \hookrightarrow (e, e, \ldots, g_i, e, e, \ldots, e)$$
$$g_i \longrightarrow$$

2. Each G_i is normal in G

3. $i \neq j \Rightarrow$ elements of G_i commute with elts of G_j

4. $Z(G_1 \times G_2 \times \ldots \times G_n) \cong Z(G_1) \times Z(G_2) \times \ldots \times Z(G_n)$

5. $(G_1 \times G_2 \times \ldots \times G_n)' \cong G_1' \times G_2' \times \ldots \times G_n'$

6. order $(g_1, g_2, \ldots, g_n) = \text{lcm}\,[\text{order}(g_i) \mid i = 1, \ldots, n]$

Examples

1. $\underbrace{\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z} \times \dots \times \mathbb{Z}/p\mathbb{Z}}_{n \text{ times}}$

elementary abelian p-group of rank n.

2. $C_m \times C_n \cong C_{mn}$ iff $(m,n)=1$

e.g. $C_3 \times C_5 \cong C_{15}$

$C_4 \times C_2 \cong C_8$

$C_3 \times C_{18} \cong C_3 \times C_2 \times C_9$

$\cong C_6 \times C_9$

3. $G_1 \times G_2 \cong G_2 \times G_1$

<u>corollary</u> $G_1 \times \dots \times G_n \cong G_{\sigma(1)} \times \dots \times G_{\sigma(n)}$ $\forall \sigma \in S_n$

4. $GL_n(k) \times GL_m(k) \cong \left\{ \begin{array}{c c} & \begin{array}{cc} n & m \end{array} \\ \begin{array}{c} n \\ m \end{array} & \left(\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array} \right) \end{array} \in GL_{n+m}(k) \right\}$

5. $S_m \times S_k \leq S_{m+k}$

<u>Warning</u>: Given $H_1 \leq G_1$, $H_2 \leq G_2$, then certainly $H_1 \times H_2 \leq G_1 \times G_2$ in a natural way. However in general $G_1 \times G_2$ has many more subgroups.

<u>Example</u>

$E_{p^2} \cong \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ has p^2 elements p^2-1 of order p

Thus there are $\frac{p^2-1}{p-1} = p+1$ subgroups of size p

$p=2$ $V \cong \{(0,0), (0,1), (1,0), (1,1)\}$ under $+$ mod 2.

3 subgroups of size 2

$\{(0,0), (1,1)\} \leq V$ is <u>not</u> of the form $H_1 \times H_2$

Recall

<u>Theorem</u> Suppose $H \trianglelefteq G$, $K \trianglelefteq G$, $G = HK$, $H \cap K = \{1\}$!

Then $G \cong H \times K$ via $(h,k) \longrightarrow hk$.

<u>Rmk</u> each G is <u>uniquely</u> of the form hk since $H \cap K = 1$.

Called <u>internal direct product</u>.

Central Products

Suppose $Z_1 \leq Z(G_1)$, $Z_2 \leq Z(G_2)$ and
$x_i \to y_i$ is an $\cong$ from Z_1 to Z_2.

Define Let $G_1 * G_2 = G_1 \times G_2 / Z$ where $Z = \{(x_i, y_i^{-1}) \mid x_i \in Z_1\}$.

Claim 1. $Z \trianglelefteq G_1 \times G_2$, actually $Z \leq Z(G_1 \times G_2)$

2. $G_1 * G_2$ has subgroups $\cong$ to G_1 and G_2 w/ intersection $\cong Z_1$.

Rmk 1. Think of $G_1 * G_2$ as "glued along Z_1"
2. $G_1 * G_2$ is ambiguous depends on Z_1 and $\cong Z_1 \to Z_2$.

Example $\mathbb{Z}_4 * D_8$ identify x^2 & r^2
$\mathbb{Z}_4 = \{1, x, x^2, x^3\}$

$\mathbb{Z}_4 * Q_8$ identify x^2 & -1.

Claim $\mathbb{Z}_4 * D_8 \cong \mathbb{Z}_4 * Q_8$

Semi Direct Products

Situation

$H \trianglelefteq G$, $K \le G$, $H \cap K = 1$. Thus $HK \le G$ but perhaps H & K don't commute.

Lemma Every elt of HK is uniquely hk

Proof $h_1 k_1 = h_2 k_2 \Rightarrow h_2^{-1} h_1 = k_2 k_1^{-1} = e$

Question How to multiply?

$$h_1 k_1 \cdot h_2 k_2 = h_1 k_1 h_2 k_1^{-1} k_1 k_2$$

$$= h_1 \, {}^{k_1}h_2 \, k_1 k_2$$

Think instead of HK as ordered pairs

$$(h_1, k_1) * (h_2, k_2) = (h_1 \, {}^{k_1}h_2, \; k_1 k_2).$$

Suppose we start with K & H, how to build this group?

What should ${}^{k_1}h_2$ be?