10/12/06

Recall Suppose $H \trianglelefteq G$, $K \leq G$, $H \cap K = 1$, $G = HK$. Then:

Prop 1. Every element of G is of the form hk in a unique way

2. Thus $G = H \times K$ as sets.

3. Under this bijection multiplication in G corresponds to:

$$(h_1, k_1)(h_2, k_2) = (h_1 {}^{k_1}h_2, k_1 k_2) = (h_1 k_1 h_2 k_1^{-1}, k_1 k_2)$$

Rmk • We have map $\varphi : K \to \operatorname{Aut} H$ $\quad \varphi(k)(h) = khk^{-1}$

• $\varphi(k)$ is not necessarily inner

Now we will start with $H \curvearrowleft K$ and build G.

Thm Let $\varphi : K \to \operatorname{Aut}(H)$ be a homomorphism. Let $k \cdot h$ denote $\varphi(k)(h)$. Let G be the set $H \times K$ and define:

$$(h_1, k_1)(h_2, k_2) = (h_1 k_1 \cdot h_2, k_1 k_2) \quad \text{Then:}$$

1. G is a group of order $|H||K|$

2. $\{(h,1) \mid h \in H\}$ and $\{(1,k) \mid k \in K\}$ are $\cong$ to H & K respectively.

3. $H \trianglelefteq G$

4. $H \cap K = 1$

5. $\forall h \in H, k \in K$, $khk^{-1} = k \cdot h$

G is a semidirect product $H \rtimes_\varphi K$ or $H \rtimes K$.

Proof

- Associativity follows from φ being a homomorphism and $\varphi(k)$ being an automorphism.
- $(1,1)$ is identity
- $(h,k)^{-1} = (k^{-1}\cdot h^{-1}, k^{-1})$

$$(h,k)(k^{-1}\cdot h^{-1}, k^{-1}) = (h\, k\cdot k^{-1}\cdot h^{-1}, kk^{-1}) = (e,e)$$

$$(k^{-1}\cdot h^{-1}, k^{-1})(h,k) = (k^{-1}\cdot h^{-1}k^{-1}\cdot h, e) = (e,e)$$

etc...

Rmk This generalizes $H \times K$:

Prop TFAE

1. $H \rtimes K \cong H \times K$ (via obvious "identity" map)
2. $\varphi : K \to \operatorname{Aut} H$ is trivial
3. $K \trianglelefteq H \cap K$

Examples

1. Suppose H is abelian. Then $\varphi(h) = h^{-1}$ is an automorphism of order 2 in $\operatorname{Aut} H$. Let $K \cong \mathbb{Z}_2 = \langle x \rangle$ with $x \cdot h = h^{-1}$.

e.g. $H = \mathbb{Z}_n = \langle r \rangle$. Then $\mathbb{Z}_n \rtimes \mathbb{Z}_2 = \langle r, x \mid r^n = e, x^2 = e, xrx^{-1} = r^{-1} \rangle \cong D_{2n}$

2. $H = \mathbb{Z}_3$, $K = \mathbb{Z}_4 = \langle x \rangle$, $x \cdot h = h^{-1}$ (so $x^2 \cdot h = h$, i.e. $x^2 h x^{-2} = h$, $x^2 \in Z(G)$)

$G = \langle h, x \mid h^3 = x^4 = e,\ xhx^{-1} = h^{-1} \rangle$ nonabelian order 12

not $\cong D_{12}$ or A_4

3 $H = \langle h \rangle \cong \mathbb{Z}_{2^n}$ $\quad K = \langle x \rangle \cong \mathbb{Z}_4$ $\quad x \cdot h = x^{-1}$ so $x^2 \in Z(G)$ as before.

Let $z = h^{2^{n-1}}$ the unique elt of order 2 in H. Then
$xzx^{-1} = z^{-1} = z$ Thus $z \in Z(G)$

Thus

$x^2 z$ is an element of order 2 in $Z(G)$!

<u>Def.</u> $\bar{G} = G/\langle x^2 z \rangle$ $\quad |\bar{G}| = 2^n \cdot 4/2 = 2^{n+1}$

$\bar{G}$ a.k.a. $Q_{2^{n+1}} = \langle h, x \mid h^{2^n} = x^4 = 1,\ xhx^{-1} = h^{-1},\ h^{2^{n-1}} = x^2 \rangle$

Generalized quaternion group.

$n=2$ $\quad Q_8 = \langle h, x \mid h^4 = x^4 = 1,\ xhx^{-1} = h^{-1},\ h^2 = x^2 \rangle$

$h \to i,\ x \to j$

4. Maximum size of Aut H:

$K = \operatorname{Aut} H$

<u>holomorph of H</u> is $H \rtimes \operatorname{Aut} H$

example: $H = \mathbb{Z}_2 \times \mathbb{Z}_2$
$\operatorname{Aut} H = GL_2(\mathbb{F}_2) \cong S_3$

$H \rtimes \operatorname{Aut} H \cong (\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes S_3 \cong S_4$

HW

5. <u>Nonabelian groups of order p^3</u>

Let $H = \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ $\quad \mathrm{Aut}\, H \cong GL_2(\mathbb{F}_p)$ order $(p^2-1)(p^2-p)$

has subgroup of order p,

e.g. $<\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}>$

$e_1 \to e_1 + e_2$

$e_2 \to e_2$

Let $K = \mathbb{Z}_p \hookrightarrow \mathrm{Aut}\, H$

Thus

$H = \langle a \rangle \times \langle b \rangle \quad K = \langle x \rangle \quad x \cdot a = ab \quad x \cdot b = b$

$G = \langle x, a, b \mid x^p = a^p = b^p = 1,\ ab = ba,\ xax^{-1} = ab,\ xbx^{-1} = b \rangle$

<u>Heisenberg group</u> $\quad$ exponent p

ex $p = 2$ $\quad D_8$

<u>Case 2</u> $\quad H = \mathbb{Z}_{p^2} \quad \mathrm{Aut}\, H \cong \mathbb{Z}_{p(p-1)}$ so $\mathbb{Z}_p \hookrightarrow \mathrm{Aut}\, H$

If $H = \langle y \rangle$ $K = \langle x \rangle$ then $x \cdot y = y^{1+p}$

$G = \langle x, y \mid x^p = y^{p^2} = 1,\ xyx^{-1} = y^{1+p} \rangle$

Exponent p^2

$p = 2$ gives Q_8

Recognizing Semidirect Products

Thm Let $H \trianglelefteq G$, $K \leq G$, $H \cap K = 1$ $G = HK$
Then $G \cong H \rtimes K$.

Rmk $K \cong G/H$.

Define Let $H \leq G$ $K \leq G$ is a complement for H if
$G = HK$ and $H \cap K = 1$.

Rmk When $H \trianglelefteq G$ and H has a complement
then G is a semidirect product.

Classification Thms

- Classify groups of order $2006 = 2 \cdot 17 \cdot 59$
- Groups of order pq