p.185 #10, 9, 15

10/19/06

Recall Given groups H & K with $\varphi: K \to \mathrm{Aut}\, H$, we get a group $H \rtimes_\varphi K$.

elements $H \times K$

operation: $(h_1,k_1)(h_2,k_2) = (h_1 k_1 \cdot h_2, k_1 k_2)$ where $k \cdot h := \varphi(k)(h)$.

Rmks 1. $H \trianglelefteq H \rtimes K$, $K \leq H \rtimes K$, $H \rtimes K = HK$, $H \cap K = 1$ where

$H = \{(h,e)\}$, $K = \{(e,k)\}$

- This characterizes semidirect products internally as well.

2. $(h,k)^{-1} = (k^{-1} \cdot h^{-1}, k^{-1})$

3. $khk^{-1} = k \cdot h$, thus action of K on H becomes conjugation inside $H \rtimes K$.

Example Classify All Groups of order $56 = 2^3 \cdot 7$.

Abelian ones:
$$\mathbb{Z}_8 \times \mathbb{Z}_7 \cong \mathbb{Z}_{56} = G_1$$
$$\mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_7 \cong \mathbb{Z}_2 \times \mathbb{Z}_{28} = G_2$$
$$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_7 \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{14} = G_3$$

Recall from Exam #1 that either the Sylow 2 or Sylow 7 is normal.

<u>Case 1</u> $C_7 \triangleleft G$. Since Sylow 2-subgroups exist we know $G \cong C_7 \rtimes P$ where $|P| = 8$. Thus for each P of order 8 we must consider maps $P \to \mathrm{Aut}\, C_7 \cong C_6$. Also if P is abelian then φ is not trivial.

<u>Lemma</u> $|P| = 8$, $\varphi: P \to \mathrm{Aut}\, C_7$ then $\ker \varphi = P$ or $|\ker \varphi| = 4$.

<u>1a</u> $P \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Choose y, z s.t. $\langle y, z \rangle = \ker \varphi \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.
Let $P = \langle x, y, z \rangle$ and $C_7 = \langle g \rangle$. Then

$$G_4 = \langle x, y, z, g \mid x^2 = y^2 = z^2 = g^7 = e,\ xy = yx,\ xz = zx,\ yz = zy,\ yg = gy,\ zg = gz,\ xgx^{-1} = g^{-1} \rangle$$

1b. $P \cong \mathbb{Z}_4 \times \mathbb{Z}_2 = \langle x, y \mid x^4 = y^2 = e,\ xy = yx \rangle$

Either $\ker \varphi = \langle x \rangle \cong \mathbb{Z}_4$ or $\ker \varphi = \langle x^2, y \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$

$$\ker \varphi = \langle x \rangle \Rightarrow G_5 = \langle x, y, g \mid x^4 = y^2 = g^7 = e,\ xy = yx,\ xg = gx,\ ygy^{-1} = g^{-1} \rangle$$

$$\ker \varphi = \langle x^2, y \rangle \quad G_6 = \langle x, y, g \mid x^4 = y^2 = g^7 = e,\ xy = yx,\ yg = gy,\ xgx^{-1} = g^{-1} \rangle$$

1c. $P \cong \mathbb{Z}_8 = \langle x \rangle$ $\ker \varphi = \langle x^2 \rangle$

$$G_7 = \langle x, g \mid x^8 = g^7 = e,\ xgx^{-1} = g^{-1} \rangle$$

Now we consider nonabelian $P \cong D_8$ or Q_8, φ may be trivial now.

1d. $P \cong Q_8$. If φ is trivial then

$$G_8 \cong C_7 \times Q_8$$

Else $|\ker\varphi| = 4$ so is $\langle i\rangle, \langle j\rangle$ or $\langle k\rangle$. All give same group

$$G_9 = \langle g, Q_8 \mid g^7 = e,\ ig = gi,\ jg = g^{-1}j \rangle$$

1e. $P \cong D_8$

$$\varphi \text{ trivial} \Rightarrow G_{10} \cong C_7 \times D_8$$

$\ker\varphi = \langle r\rangle \cong Z_4$

$$G_{11} = \langle g, D_8 \mid g^7 = e,\ gr = rg,\ gs = sg^{-1} \rangle$$

$\ker\varphi = \langle s, r^2\rangle$

$$G_{12} = \langle g, D_8 \mid g^7 = e,\ gs = sg,\ rg = g^{-1}r \rangle$$

<u>Case 3</u> Let $P \in Syl_2(G)$, $P \trianglelefteq G$. So $\varphi: C_7 \to \mathrm{Aut}\, P$

C_7 acts by conjugation on P. Orbits size 1 or 7. Thus all elts of $P - \{e\}$ are conjugate so same order so $P = Z_2 \times Z_2 \times Z_2$.

So $(Z_2 \times Z_2 \times Z_2) \rtimes C_7$

Thus we need elements of order 7 in $\mathrm{Aut}(Z_2^3) = GL_3(F_2)$ order $\underset{7 \cdot 2^3 \cdot 3}{\overset{}{168}}$

Thus they exist and are all conjugate, hence exactly one such group. $G_{13} \cong Z_2^3 \rtimes Z_7$

Example

Wreath product

Suppose $K \le S_n$, L a group. Let $H = \underbrace{L \times L \times \dots \times L}_{n \text{ times}}$

Then $K \hookrightarrow \operatorname{Aut} H$ in obvious way, for $\sigma \in K$ let

$$\sigma(h_1, h_2, \dots, h_n) = (h_{\sigma(1)}, h_{\sigma(2)}, \dots, h_{\sigma(n)}).$$

Thus

$$(L \times L \times \dots \times L) \rtimes K, \text{ called wreath product } L \wr K.$$

Rmk Suppose K any group, then $K \hookrightarrow \operatorname{Sym}(K)$ (Cayley's Thm) so

$$H^{\times |K|} \rtimes K \text{ still exists}$$

Example Sylows of S_n.

$$C_3 \wr C_3 = \langle (123), (456), (789), (147)(258)(369) \rangle = \operatorname{Syl}_3(S_9)$$