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Chapter 6 – p-groups, nilpotent groups, solvable groups

Def. $M < G$ is a **maximal subgroup** if $\nexists$ $M < H < G$.

- Finite groups every subgroup is in some maximal subgroup
- $G = \mathbb{Z}$ $M = p\mathbb{Z}$
- $\mathbb{Q}, +$ no maximal subgroups
- G simple, lots of research

Mega p-group Theorems

Let $|P| = p^q$.

1. $Z(P) \neq \{e\}$
2. $H \trianglelefteq P$, $H \neq \{e\} \Rightarrow H \cap Z(P) \neq e$
3. Suppose $H \trianglelefteq P$, then H contains a normal subgroup of P of every possible order. In particular P contains normal subgroups of order $1, p, p^2, p^3, \ldots, p^q$
4. $H < P \Rightarrow H < N_p(H)$
5. Every maximal subgroup is normal of index p.

lots of normal subgroups!

Proof

2. $H \trianglelefteq P \Rightarrow$ H is a union of conjugacy classes, all have order a power of p.
   Thus at least $p-1$ nonidentity classes of size 1, i.e. central elements

1. Set $H = P$ in #2

3. Induce on a.

- $H \cap Z(P) \neq e$, now look at $H / H \cap Z(P)$.
- Need to use true for abelian groups

4. Induction again. WLOG $Z(P) \leq H$. Now look at $\bar{G} = G/Z(P)$
$\bar{H} = H/Z(P)$

$\bar{H} < N_{\bar{G}}(\bar{H})$, use corr. thm.

5. Let M be maximal $M < N_G(M)$ so $M \trianglelefteq G$, now use earlier Thm

<u>Central Series</u> G a group

$Z_0(G) = \{1\}$ $Z_1(G) = Z(G)$

<u>Define</u> $Z_2(G)$ so that $Z_2(G) \leftrightarrow Z(G/Z(G))$ under Lattice Isomorphism Thm

Inductively Define $Z_{i+1}(G)$ so $Z_{i+1}/Z_i = Z(G/Z_i)$

<u>Def</u> $1 \leq Z(G) \leq Z_2(G) \leq \dots$ is ~~lower~~ upper <u>central series</u>

<u>Rmks</u>

1. $Z_i(G)$ char G $\forall i$
2. If at any stage $Z(G/Z_i) = \{1\}$ then $Z_{i+1} = Z_i$ and series is constant after that.
3. If G/Z_i is abelian then $Z_{i+1} = G$.

Def. G is <u>nilpotent</u> if $Z_c(G) = G$ some c. Smallest such c is called <u>nilpotence class</u> of G

<u>Rmks</u>

1. G is abelian iff G is nilpotent of class 1

2. Z_{i+1}/Z_i is abelian. Thus nilpotent $\Rightarrow$ solvable.

   abelian $\subset$ nilpotent $\subset$ solvable

3. Finite p-groups are nilpotent by omnibus thm.
   In particular if $|P| = p^a$ $a > 1$ then P is class $\leq a-1$.
   (use $G/Z(G)$ cyclic $\Rightarrow$ G abelian)

Alternate characterizations of finite nilpotent groups:

<u>Thm</u> Let $|G| = p_1^{a_1} p_2^{a_2} \cdots p_s^{a_s}$, $P_i \in Syl_{p_i}(G)$. TFAE:

1. G is nilpotent

2. $H < G \Rightarrow H < N_G(H)$

3. $P_i \trianglelefteq G$ $\forall i$

4. $G \cong P_1 \times P_2 \times \ldots \times P_s$

* i.e. Finite nilpotent groups $\longleftrightarrow$ direct product of p-groups

Proof

1→2 G nilpotent ⇒ $G/Z(G)$ nilpotent so induce.

2→3 Let $P=P_i$, $N=N_G(P)$, so P char $N \trianglelefteq N_G(N)$.
Thus $N_G(N)=N$. Thus $N=G$ by 2
so $P \trianglelefteq G$.

3→4 Done already

4→1 $Z(P_1 \times \dots \times P_s) = Z(P_1) \times \dots \times Z(P_s)$ etc.

Frattini Argument $H \trianglelefteq G$, G finite, $P \in Syl_p(H)$. Then
$G = H N_G(P)$ and $|G:H| \mid |N_G(P)|$

Proof

Let $g \in G$. $gPg^{-1} \leq gHg^{-1} = H$ so
$gPg^{-1} = hPh^{-1}$ $h \in H$
so $h^{-1}g \in N_G(P)$
$g \in H N_G(P)$

$|G:H| = HN_G(P)/H = N_G(P)/H \cap N_G(P) \mid |N_G(P)|$

Corollary G is nilpotent iff every maximal subgroup is ~~nilpotent~~. normal.

Proof ⇒ M maximal, $M < N_G(M) \Rightarrow M \trianglelefteq G$

⇐ Let $P \in Syl_p(G)$. ETS $P \trianglelefteq G$. If not enlarge ~~$P \leq M \leq$~~ G. $P < N_G(P) < M < G$

Then $M \trianglelefteq G$ so $G = M N_G(P)$ ⨳