p.198 #21, 22a., 24, 25, 26

Recall   G a group, upper central series:

$$Z_0(G)=1,\ Z_1(G)=Z(G) \ldots \quad Z_i(G)/Z_{i-1}(G) \cong Z(G/Z_{i-1}(G)).$$

Def. G is nilpotent of class c if $Z_c(G)=G$, $Z_{c-1}(G)\neq G$.

Equivalent Conditions for nilpotence

1. $H<G \Rightarrow H<N_G(H)$
2. Sylow subgroups are all normal
3. G is a direct product of p-groups
4. Maximal subgroups are all normal.
5. Elements of coprime order commute.
6. G has a normal subgroup of every possible order.

Def. Let $H,K\leq G$. $[H,K]=\langle [h,k]\rangle$

$G'=[G,G]$ derived (commutator) subgroup.

Def. $G^0=G \quad G^1=[G,G] \quad G^k=[G,G^{k-1}]$

Thm G is nilpotent of class c iff $G^c=e$, $G^{c-1}\neq e$

Proof show $G^{c-i}<Z_i(G)$ by induction on c.

**** Not same Series***

$$G=S_3 \quad G^1=A_3 \quad G^2=[S_3,A_3]=G^1$$

$$Z_0=\{e\} \quad Z_1=Z(S_3)=\{e\}$$

Even if nilpotent, may not be same series

Solvable Groups

Def $G = G$, $G^{(1)} = G' = [G, G]$, ... $G^{(i+1)} = [G^{(i)}, G^{(i)}]$
called derived series.

Thm G is solvable iff only if $G^{(n)} = 1$ some n

Proof $\Leftarrow$ trivial, $G^{(i+1)}/G^{(i)}$ is abelian

$\Rightarrow$ Assume
$1 = H_0 \trianglelefteq H_1 \trianglelefteq \cdots \trianglelefteq H_s = G$ w/ H_i/H_{i-1} abelian

Claim $G^{(i)} \leq H_{s-i}$

Proof $i = 0$ ✓
Assume $G^{(i)} \leq H_{s-i}$

Then H_{s-i}/H_{s-i-1} is abelian. Thus

$$(H_{s-i})' \leq H_{s-i-1}$$
$$\begin{array}{c} (H_{s-i})' \\ \| \\ [H_{s-i}, H_{s-i}] \\ \cup \\ [G^{(i)}, G^{(i)}] = G^{(i+1)} \end{array}$$
✓

0. Subgroups of solvable

Thm 1. Homomorphic images of solvable groups are solvable.

2. $N \trianglelefteq G$, G/N, N solv $\Rightarrow$ G is

Rmk Good for inductive proofs

Much more deep results...

Thm

1. Burnside's $p^a q^b$ theorem $|G| = p^a q^b \Rightarrow G$ solvable

2. (P. Hall) Suppose $\forall p \,|\, |G|$ that G has a subgroup of order m where $|G| = p^a m$. Then G is solvable. (generalizes Burnside)

3. (Feit-Thompson) $|G|$ odd $\Rightarrow G$ solvable (255 pages)

4. (Thompson) $\forall x, y \in G$ $\langle x, y \rangle$ solvable $\Rightarrow G$ is

FREE GROUP.

Free Groups

- start w/ a set S
- make a group w/ S and no relations
- universal property.

Construction

1. Start w/ a set S
2. Let S^{-1} be a disjoint set w/ a bijection $s \longleftrightarrow s^{-1}$
3. Throw in $1 = 1^{-1}$

Def. A *word* is a sequence $(s_1, s_2, s_3, \dots)$ with $s_i \in S \cup S^{-1} \cup 1$, $s_i = 1$ $\forall i \gg N$.

Def. Word is *reduced* if
- No $s_{i+1} = s_i^{-1}$
- $s_k = 1 \Rightarrow s_\ell = 1 \ \forall \ell > k$

Can write $w = (s_1^{\epsilon_1} s_2^{\epsilon_2} \dots s_n^{\epsilon_n} 1 1 1 \dots)$ $\quad \epsilon_i = \pm 1$

Operation concatenate and cancel.

Thm Let $F(S)$ be the set of reduced words under this operation. Then $F(S)$ is a group.

Thm Let G be a group, $\varphi: S \to G$ a map $\exists!\ \Phi: F(S) \to G$

st

$$\begin{tikzcd} S \arrow[r, hook] \arrow[dr, "\varphi"'] & F(S) \arrow[d, "\Phi"] \\ & G \end{tikzcd}$$

commutes.

COR $F(S)$ unique.

- generators, rank

<u>Schreier Thm</u> Subgroups of free groups are free.

Presentations

- f.g.
- f.p.