

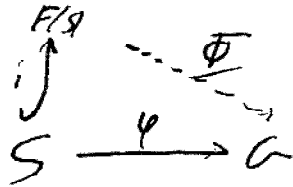
10/31/06

Recall: Set S . $F(S)$ = free group on S .

- elements = reduced words, $S \subseteq F(S)$.
- operation is concatenation then cancellations.

The (Universal Property) Let G be any group, S a set and $\psi: S \rightarrow G$. Then ψ extends to a unique group homomorphism $\Phi: F(S) \rightarrow G$ such that

$$\Phi(s) = \psi(s) \quad \forall s \in S$$

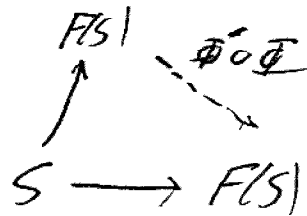


Cor Suppose $S \xrightarrow{i} F$ F another group w/ this property. Then $F \cong F(S)$.

Proof



Then consider



$S \rightarrow F(S)$
commutes thus

$$\Phi \circ i = \text{Id}$$

by uniqueness

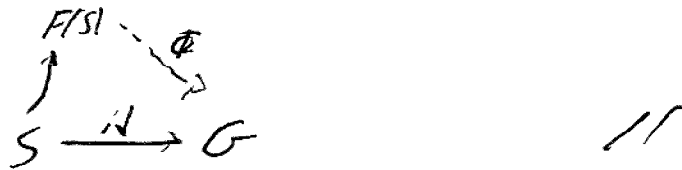
Ranks

A group is free of rank n if it is \cong to FS_n for some S of cardinality n .

Exercise FS_n is determined up to \cong by cardinality of S .

Prop Every group is the homomorphic image of a free group.

Proof Let $S = G$ with $\nu = \text{id}$.



Rank Any set of generators for G will suffice.

Def Let $S \subseteq G$ be generators, $G = \langle S \rangle$.

1. A presentation for G is a pair (S, R) where

$\pi: FS_n \rightarrow G$ has kernel $\langle R \rangle^{FS_n}$

S are generators, R are relations

2. Finitely generated means $|S| < \infty$

Finitely presented means $|S|, |R| < \infty$

Warning Kernel is normal closure of R

Examples

1. $|G| < \infty \Rightarrow G$ is f.p.

Proof If $g_i g_j = g_k$ then include $g_i g_j g_k^{-1}$ in R .

2. Abelian groups

$$\mathbb{Z}_8 \times \mathbb{Z}_{10} = \langle x, y \mid [x, y] = e, x^8 = y^{10} = e \rangle$$

Remark Hard to study group given presentation

3. Braid group

$$B_n = \langle \sigma_1, \sigma_2, \dots, \sigma_{n-1} \mid \begin{array}{l} \sigma_i \sigma_j = \sigma_j \sigma_i \quad |i-j| > 1 \\ \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \end{array} \rangle$$

Hard Theorem Subgroup of free group

Remark Suppose $|S| \geq 2$. Then $F(S)$ has subgroups of all countable ranks

Remark Word Problems

Rings

- Def.
- Ring w/ identity
- Div. Ring (skew field)

Examples

1. Any abelian group, 0 multiplication.
2. Integers
3. $\mathbb{Z}/n\mathbb{Z}$
4. real quaternions
* division ring $(a+bi+cj+dk)^{-1} = \frac{a-bi-cj-dk}{a^2+b^2+c^2+d^2}$
5. Polynomial rings
6. Ring of functions from a set to a ring
7. matrices, etc.,

Defs

- zero divisors
- units, group of units
- integral domain

Prop Cancellation in integral domain

Thm Finite integral domains are fields

Example $\mathbb{Q}(\sqrt{D})$ where D is squarefree WLOG.

check this is a field.

Example $\mathbb{Z}[\sqrt{D}]$

subrings, etc.