10/31/06

Recall: Set S. $F(S)$ = free group on S.

- elements = reduced words, $S \subseteq F(S)$.
- operation is concatenation then cancellations.

Thm (Universal Property) Let G be any group, S a set and $\varphi: S \to G$. Then φ extends to a unique group homomorphism $\Phi: F(S) \to G$ such that

$$\Phi(s) = \varphi(s) \quad \forall s \in S$$

$$\begin{tikzcd} F(S) \arrow[dr, dashed, "\Phi"] & \\ S \arrow[u, hook, "i"] \arrow[r, "\varphi"] & G \end{tikzcd}$$

Cor Suppose $S \xhookrightarrow{i'} F$, F another group w/ this property. Then $F \cong F(S)$.

Proof

$$\begin{tikzcd} F(S) \arrow[dr, dashed, "\Phi"] & \\ S \arrow[u, hook, "i"] \arrow[r, "i'"] & F \end{tikzcd} \qquad \begin{tikzcd} F \arrow[dr, dashed, "\Phi'"] & \\ S \arrow[u, hook, "i'"] \arrow[r, "i"] & F(S) \end{tikzcd}$$

Then consider

$$\begin{tikzcd} F(S) \arrow[dr, dashed, "\Phi' \circ \Phi"] & \\ S \arrow[u] \arrow[r] & F(S) \end{tikzcd}$$

$s \mapsto s$

commutes. Thus $\Phi' \circ \Phi = Id$ by uniqueness.

Rmks

A group is free of rank ? if it is $\cong$ to $F(S)$ for some S of cardinality ?

Exercise $F(S)$ is determined up to $\cong$ by cardinality of S.

Prop Every group is the homomorphic image of a free group.

Proof Let $S = G$ with $\varphi = id$.

$$\begin{tikzcd} F(S) \arrow[dr, dashed, "\Phi"] & \\ S \arrow[u] \arrow[r, "id"] & G \end{tikzcd}$$

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Rmk Any set of generators for G will suffice.

Def Let $S \subseteq G$ be generators, $G = \langle S \rangle$.

1. A presentation for G is a pair (S, R) where

   $\pi : F(S) \to G$ has kernel $\langle R \rangle^{F(S)}$

   S are generators, R are relations

2. Finitely generated means $|S| < \infty$

   Finitely presented means $|S|, |R| < \infty$

Warning Kernel is normal closure of R

Examples

1. $|G| < \infty \Rightarrow G$ is f. p.

   Proof If $g_i g_j = g_k$ then include $g_i g_j g_k^{-1}$ in R.

2. Abelian groups

$$\mathbb{Z}_8 \times \mathbb{Z}_{10} = \langle x, y \mid [x,y] = e,\ x^8 = y^{10} = e \rangle$$

Rmk Hard to study group given presentation

3. Braid group

$$B_n = \langle \sigma_1, \sigma_2, \ldots, \sigma_{n-1} \mid \sigma_i \sigma_j = \sigma_j \sigma_i \quad |i-j| > 1,\quad \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \rangle$$

Hard Thm Subgroup of free group

Rmk Suppose $|S| \geq 2$. Then $F(S)$ has subgroups of all countable ranks

Rmk Word Problem

Rings

- Def.
- Ring w/ identities
- Div. Ring (skew field)

Examples

1. Any abelian group, 0 multiplication
2. Integers
3. $\mathbb{Z}/n\mathbb{Z}$
4. real quaternions
   * division ring $(a+bi+cj+dk)^{-1} = \dfrac{a-bi-cj-dk}{a^2+b^2+c^2+d^2}$
5. Polynomial rings
6. Ring of functions from a set to a ring
7. matrices, etc.

Defs

- zero divisor
- units, group of units
- integral domain

Prop Cancellation in integral domain

Thm Finite integral domains are fields

Example $\mathbb{Q}(\sqrt{D})$ where D is squarefree WLOG.

check this is a field

Example $\mathbb{Z}[\sqrt{D}]$

subrings, etc.