11/7/06

Recall: An ideal I is a subring such that $rI \subseteq I$, $Ir \subseteq I$ $\forall r \in R$.

Thm Let I be an ideal. Then R/I is a ring under mult

$$(r+I)(s+I) = rs + I$$

Thm ($1^{st} \cong$ Thm) Let $\varphi: R \to S$ be a ring homomorphism. Then $\ker \varphi$ is an ideal, $\varphi(R)$ is a subring, and

$$R/\ker\varphi \cong \varphi(R)$$

Furthermore if I is an ideal then $\pi: R \to R/I$, $\pi(r) = r + I$, is a ring homomorphism w/ kernel I.

* {ideals} = {kernels of homomorphisms}

Examples of ideals

1. $n\mathbb{Z} \le \mathbb{Z}$

2. Let $x \in R$. The principal ideal generated by x

$$(x) = \{ r_1 x r_1' + \dots + r_n x r_n' \}$$

smallest ideal containing x

left ideal $Rx = \{ rx \mid r \in R \}$
right ideal $xR = \{ xr \mid r \in R \}$

3. $R = M_n(F)$

$$I_j = \left\{ \begin{array}{c} \text{col } j \\ \downarrow \\ \begin{pmatrix} 0 & 0 & \cdots & \ast & 0 \\ 0 & 0 & \cdots & \ast & 0 \\ 0 & 0 & \cdots & \ast & 0 \end{pmatrix} \end{array} \right\}$$

I_j is a left ideal

$$K_i = \left\{ \begin{pmatrix} 0 & \text{---} & 0 \\ \ast & \ast \, \text{---} & \ast \\ 0 & \text{---} & 0 \\ 0 & \text{---} & 0 \end{pmatrix} \leftarrow \text{row } i \right\}$$

is a <u>right ideal</u>

<u>Thm</u> R has no nontrivial 2-sided ideals

4. Fields (or div rings) have no nontrivial ideals

5. $\mathbb{R}[x]/(x^2) = \{a + bx \mid x^2 = 0\}$

6. $\mathbb{R}[x]/(x^2+1) \cong \mathbb{C}$

<u>Def</u>. 6.5 augmentation ideal

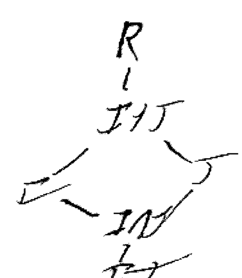
7. I, J ideals $\Rightarrow I \cap J$ is an ideal

<u>Def</u>. Let I & J be ideals of R. Their <u>sum</u>

$$I + J = \{a + b \mid a \in I, b \in J\}$$

product $IJ = \{$ finite sums of $ab \mid a \in I, b \in J\}$ (powers $I^2, I^3, \ldots$

<u>Rmk</u> $IJ \subseteq I \cap J$, not always $=$



Other $\cong$ Theorems

An ideal is a subgroup of $R,+$ so all the $\cong$ Thms hold as groups, check the $\cong$ are actually ring isom.

Second $\cong$ Thm

A a subring, B an ideal. Then $A+B$ is a subring, $A\cap B$ an ideal

$$\begin{tikzcd}[column sep=small] & R \arrow[d, dash] & \\ & A+B \arrow[dl, dash] \arrow[dr, dash] & \\ B \arrow[dr, dash] & & A \arrow[dl, dash] \\ & A\cap B & \end{tikzcd}$$

$$A+B/_B \cong A/A\cap B$$

3rd $\cong$ Thm

$$\begin{tikzcd} R \arrow[d, dash] \\ J \arrow[d, dash] \\ I \end{tikzcd}$$

Then $J/_I$ ideal in $R/_I$ and

$$R/I\Big/_{J/I} \cong R/J$$

Lattice $\cong$

$I \subset$ ideal R.

- subrings containing $I \longleftrightarrow$ subrings R/I,

  ideal $\longleftrightarrow$ ideal

More ideals.

- ideals generated by a subset
- principal / nonprincipal ideals

  e.g. $(2, x) \in \mathbb{Z}[x]$

- maximal ideals may not exist, e.g. $\mathbb{Q}$ w/ trivial $\cdot$
- prime ideals

Prop In ring w/ 1 maximal ideals exist

Zorn's lemma

Commutative rings

maximal $M \longleftrightarrow R/M$ a field

prime $P \longleftrightarrow R/M$ int domain