A 254 # 39 19,24

11/16/06

Let R be a ring w/ identity.

Def An ideal (proper) is maximal if $\nexists$ $M \subsetneq I \subsetneq R$.

Thm Assume R is commutative. Then M is maximal iff R/M is a field.

Rmk 1 Does need R to have a unit.

Rmk 2 R/M a field $\Rightarrow$ M is maximal, even w/out assumption that R is commutative.

Example $R = FG$, augmentation ideal is maximal!

Example Maximal ideals of $\mathbb{Z}$ $\longleftrightarrow$ (p) p prime

Example $\varphi: R \twoheadrightarrow S$ a surjective ring homo. If N max in S then $\varphi^{-1}\{N\}$ maximal in R.

Recall $p \in \mathbb{Z}$ is prime iff $p | ab \Rightarrow p | a$ or $p | b$.

Def Let R be commutative ring w/ 1. An ideal P is a **prime ideal** if

$$ab \in P \Rightarrow a \in P \text{ or } b \in P.$$

Ex $n\mathbb{Z}$ is prime iff n is prime so in $\mathbb{Z}$ prime = maximal.

Thm R comm w/ 1. Then an ideal P is prime iff R/P is integral domain.

COR Maximal $\Rightarrow$ prime

Examples

1. $(x) \subset \mathbb{Z}[x]$ is prime, not maximal.

2. $(0) \subset R$ is prime iff R is an integral domain.

3. $(p(x)) \subset \mathbb{Z}[x]$ is prime iff $p(x)$ is irreducible.

Some important ideals

Def I ideal in R comm w/ 1.

$$\sqrt{I} = \operatorname{rad} I = \{ r \in R \mid r^n \in I \text{ some } n > 0 \}$$
is radical of I.

Example
$$I = ((x-1)^2) \subset \mathbb{Z}[x]$$
$$\sqrt{I} = (x-1)$$

Example
Suppose $I = (0)$. Then
$$\sqrt{I} = \{\text{nilpotent elements}\}$$
called nilradical $\eta(R)$.

If $I = \sqrt{I}$ then I is a radical ideal.
e.g. any prime ideal

Exercise $I \subseteq \sqrt{I}$ and
$$\sqrt{I}/I = \eta(R/I)$$

Thus quotient by a radical ideal has no nilpotent elements.

Def. Let $I \subset R$ comm.

The <u>Jacobson radical of I</u>

$$\mathrm{Jac}(I) = \bigcap_{\substack{\text{maximal ideals} \\ \text{containing}}} I$$

Jacobson radical of R is $\mathrm{Jac}(0)$

$$\mathrm{Jac}(R) = \bigcap \text{maximal ideals}$$

← noncomm m then $\bigcap$ max left = $\bigcap$ max right

<u>Examples</u> 1. $\mathrm{Jac}(\text{field}) = 0$

2. $J(\mathbb{Z}/8\mathbb{Z}) = 2\mathbb{Z}/8\mathbb{Z}$

3. $J(B) = U^{\ulcorner}$

Def. A <u>local ring</u> is a comm ring w/ a unique maximal ideal.

<u>Thm</u> 1. If R is local w/ max ideal M then $R - M$ is all units

2. If R comm w/ 1 and {nonunits} is a max ideal then R is local.

<u>Example</u> {rationals w/ odd denominator}