11/21/06

Review

$M < S$ is a <u>maximal ideal</u> if $\nexists\ M \subsetneq I \subsetneq R$.

<u>Rmk</u> In ring w/ identity, maximal ideals always exist.

~~Thm~~

<u>Thm</u> 1. R/M a field $\Rightarrow$ M is a maximal ideal.

2. R comm ring w/ 1 then M maximal $\Rightarrow$ R/M a field.

<u>Def</u> R comm ring w/ 1. Then I is <u>prime</u> if $ab \in I \rightarrow a \in I$ or $b \in I$.

<u>Rmk</u> P can't be "factored" in ideal arithmetic.

<u>Thm</u> R comm. w/ 1. Then P is prime $\Longleftrightarrow$ R/P is an int domain.

<u>Cor</u> Comm ring w/ 1 then maximal $\Rightarrow$ prime

<u>Some special ideals</u>

<u>Commutative Rings w/ identity</u>

1. $\eta(R)$ = <u>nilradical</u> of R = $\{x \in R \mid x \text{ is nilpotent}\}$

2. More generally for I an ideal,

$$\text{rad}\, I = \sqrt{I} = \{r \mid r^n \in I \text{ some } n\}$$

Thus a. $\eta(R) = \sqrt{(0)}$

b. $\eta(R/I) = \sqrt{I}/I$

c. $\sqrt{\sqrt{I}} = 0$

3. I an ideal. Then

$$\operatorname{Jac} I = \bigcap_{\substack{\text{maximal}\\ \text{ideals}\\ M \supseteq I}} M.$$

$\operatorname{Jac} R = \bigcap_{\text{ideals}}^{\text{maximal}}$ is <u>Jacobson Radical of ring</u>.

<u>Prop</u> $\mathfrak{n}(R) \subseteq \operatorname{Jac} R$.

<u>Proof</u> Let M be a maximal ideal. Then $M \subseteq \sqrt{M} \subsetneq R$ so $M = \sqrt{M}$. Let x be nilpotent. Then

$$x^n = 0 \in M \text{ so } x \in \sqrt{M} = M.$$

Thus $\mathfrak{n}(R) \subseteq M$.

<u>Remark</u> Not always equal, in Artinian ring they are.

<u>Remark</u> ~~R/Jac R has~~ Jac R has many properties of Frattini subgroup.

<u>Local Rings</u>

<u>Def.</u> A comm. ring is <u>local</u> if it has a unique maximal ideal.

<u>Prop</u> Suppose R is comm w/1. Then R is local iff the set of nonunits is an ideal.

<u>Example</u>
- $\mathbb{F}_p P$ where $|P| = p^n$ (HW; aug ideal nilpotent, codim 1)
- Fields
- $\{\frac{a}{b} \mid a \in \mathbb{Z}\}$ b odd

Rings of Fractions

Situation: R comm ring.

Recall a not a zero divisor then $ab = ac \Rightarrow b = c$.

Goal Get a bigger ring where non-zero divisors are actually units

Example $R = \mathbb{Z}$, want $\neq 0$ elts invertible.

1. Define fraction $\frac{a}{b}$ but $\frac{a}{b} = \frac{c}{d} \leftrightarrow ad = bc$.

   Thus $\frac{a}{b} \leftrightarrow$ equiv class of ordered pairs

2. $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$ $\quad \frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bc}$

   check well defined

3. Thus we have $\mathbb{Z} \cong \{\frac{a}{1}\} \subseteq \mathbb{Q}$.

Rmk If $bd = 0$ then $d = \frac{d}{1} = \frac{bd}{b} = \frac{0}{b} = 0$
$b, d \neq$

so we get a contradiction. Thus we can't put zero divisors in denom.

<u>Thm</u> R a comm ring. $D \neq \emptyset$, $D \subseteq R$ has no zero divisors and does not contain 0. Assume D is closed under multiplication.

Then $\exists$ a comm ring w/ 1 Q such that

1. $R \subseteq Q$ as a subring
2. Elts of D are units in R
3. Every element of Q is of form $rd^{-1} \equiv \frac{r}{d}$ for $d \in D$,

   — Thus if $D = R - \{0\}$ then Q is a field

4. Universal property

   Suppose S comm ring w/ 1, $\varphi : R \xrightarrow{1\text{-}1} S$ st. $\varphi(d)$ is a unit $\forall d \in D$. Then

   $\exists !$ $\Phi : Q \to S$ Φ 1-1

$$\begin{tikzcd} Q \arrow[drr, dashed, "\Phi"] & & \\ \varphi : R \arrow[u, phantom, "\cup" rotate=90] \arrow[rr, "\varphi"] & & S \end{tikzcd}$$

i.e. Q is "~~largest~~ smallest" such ring.

<u>Proof</u>

Proof

Def 1. Q is called the ring of fractions of D w.r.t. R,

$$D^{-1}R$$

2. If R is I.D., $D = R - 0$ then
Q is field of fractions
a.k.a. quotient field of R

Rmk Generalizes to prime ideals, localization