

11/21/06

## Review

$M \subset S$  is a maximal ideal if  $\nexists M \subsetneq I \subsetneq R$ .

Prop In ring w/ identity, maximal ideals always exist.

Thm

Thm 1.  $R/M$  a field  $\Rightarrow M$  is a maximal ideal!

2.  $R$  comm ring w/ 1 then  $M$  maximal  $\Rightarrow R/M$  a field.

Def  $R$  comm ring w/ 1. Then  $I$  is prime if  $ab \in I \rightarrow a \in I$  or  $b \in I$ .

Prop  $P$  can't be "factored" in ideal arithmetic.

Thm  $R$  comm w/ 1. Then  $P$  is prime  $\iff R/P$  is an int domain.

Cor Comm ring w/ 1 then maximal  $\Rightarrow$  prime

## Some special ideals

### Commutative Rings w/ identity

1.  $\mathfrak{N}(R) = \text{nilradical of } R = \{x \in R \mid x \text{ is nilpotent}\}$

2. More generally for  $I$  an ideal,

$$\text{rad } I = \sqrt{I} = \{r \mid r^n \in I \text{ some } n\}$$

Thus a.  $\mathfrak{N}(R) = \sqrt{(0)}$

b.  $\mathfrak{N}(R/I) = \sqrt{I}/I$

c.  $\sqrt{\sqrt{I}} = I$

3 If an ideal. Then

$$\text{Jac } I = \bigcap_{\substack{\text{maximal} \\ \text{ideals} \\ M \supseteq I}} M.$$

$\text{Jac } R = \bigcap_{\text{ideals}}^{\text{maximal}}$  is Jacobson Radical of ring.

Prop  $\mathfrak{nil}(R) \subseteq \text{Jac } R$ .

Proof Let  $M$  be a maximal ideal. Then  $M \subseteq \sqrt{M} \subseteq R$  &  $M = \sqrt{M}$ . Let  $x$  be nilpotent. Then

$$x^n = 0 \in M \text{ so } x \in \sqrt{M} = M.$$

Thus  $\mathfrak{nil}(R) \subseteq M$ .

Remark Not always equal, in Artinian ring they are.

Remark  $\mathfrak{nil}(\text{Jac } R)$  has  $\text{Jac } R$  has many properties of Fitting subgroup.

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### Local Rings

Def. A comm. ring is local if it has a unique maximal ideal.

Prop Suppose  $R$  is comm w/1. Then  $R$  is local iff the set of nonunits is an ideal.

Example •  $\mathbb{F}_p[x]$  where  $|p| = p^n$  (HW, any ideal nilpotent, codim 1)  
• Fields  
•  $\{ \frac{a}{p^n} \mid a \in \mathbb{Z} \}$  local

## Rings of Fractions

Situation -  $R$  comm ring.

Recall  $a$  not a zero divisor then  $ab=ac \Rightarrow b=c$ .

Goal Get a bigger ring where non-zero divisors are actually units.

Example  $R = \mathbb{Z}$ , want  $\neq 0$  elts invertible.

1. Define fraction  $\frac{a}{b}$  but  $\frac{a}{b} = \frac{c}{d} \Leftrightarrow ad=bc$ .

Thus  $\frac{a}{b} \Leftrightarrow$  equiv class of ordered pairs.

$$2. \frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd} \quad \frac{a}{b} - \frac{c}{d} = \frac{ad-bc}{bd}$$

check well defined

3. Thus we have  $\mathbb{Z} \cong \left\{ \frac{a}{1} \right\} \subseteq \mathbb{Q}$ .

Remark IF  $bd=0$  then  $d = \frac{1}{1} = \frac{bd}{b} = \frac{0}{b} = 0$   
 $b, d \neq$

so we get a contradiction  
 zero divisors in denom.

Thus we can't put

Thm  $R$  a comm ring,  $D \neq \emptyset$ ,  $D \subseteq R$  has no zero divisors and does not contain 0. Assume  $D$  is closed under multiplication.

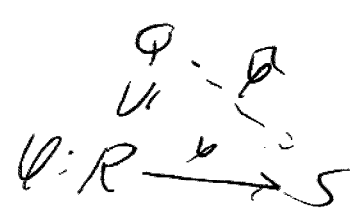
Then  $\exists$  a comm ring w/ 1  $Q$  such that

- 1.  $R \subseteq Q$  as a subring
- 2. Elts of  $D$  are units in  $R$
- 3. Every element of  $Q$  is of form  $rd^{-1} \equiv \frac{r}{d}$  for  $d \in D$ ,  
 - Thus if  $D = R - \{0\}$  then  $Q$  is a field

4. Universal property

Suppose  $S$  comm ring w/ 1,  $\psi: R \rightarrow S$  st.  $\psi(d)$  is a unit  $\forall d \in D$ . Then

$$\exists \text{ @ } \psi: Q \rightarrow S \quad \psi|_R = \psi$$



i.e.  $Q$  is "<sup>smallest</sup>/<sub>largest</sub>" such an

Proof  
Proof

Def. 1.  $Q$  is called the ring of fractions of  $D$  unit.  $R$ ,  
 $D^{-1}R$

2. If  $R$  is I.D.,  $D=R \setminus 0$  then

$Q$  is field of fractions

a.k.a. quotient field of  $R$ .

RMK Generalizes to prime ideals, localization