Math 8300-Final Exam December 11, 2006

Part I: Do all six problems.

1. Suppose $H \leq G$ is the unique subgroup of order $|H|$. Prove H is normal.

2. a. Give the conjugacy classes of S_3 and D_8.

   b. Write down a set of generators for a Sylow-3 subgroup of S_9.

3. How many elements of order 7 must there be in a simple group of order 168?

4. Give an example of:

   a. A commutative ring that is not an integral domain.

   b. A Euclidean domain that is not a field.

   c. An integral domain that is not a UFD.

   d. A maximal ideal in $\mathbb{Z}[x]$.

5. Define *solvable group.*

6. Explain in detail the construction of a semidirect product of groups.

Part II: Choose exactly three problems.

1. Let R be a ring with the property that every ideal is finitely generated. Let $I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$ be an ascending chain of ideals. Prove that the chain terminates, i.e. $\exists r > 0$ such that $I_r = I_{r+1} = \cdots$.

2. Let R be a commutative ring. Let I and J be ideals and assume P is a prime ideal that contains IJ. Prove either I or J is contained in P.

3. Prove that in a PID, an ideal is prime if and only if it is maximal.

4. Let I be an ideal of a ring R with identity. Prove R/I is a field implies I is maximal.

5. Let R be a commutative ring. Define a greatest common divisor of two elements. Prove that in a PID, gcd's always exist.

Part III: Choose exactly three problems.

1. Let N be a normal subgroup of a finite group G and suppose G/N is a p-group for some prime p. Further suppose that $N \leq Z(G)$. Prove that the commutator subgroup G' is a p-group.

2. Suppose a group G has a conjugacy class with exactly two elements. Prove G cannot be simple.

3. Let G be a finite nilpotent group and $\{e\} \neq N \trianglelefteq G$. Prove $N \cap Z(G) \neq \{e\}$.

4. Classify all groups of order $2006 = 2 \cdot 17 \cdot 59$.

5. Prove that an infinite simple group cannot have a subgroup of finite index.