

Math 8300 Fall 2006 Midterm Exam #1.

Instructions: Choose five out of the seven problems. Be sure to clearly indicate which five you want graded.

1. Suppose $H \leq G$. Prove that $\text{Aut}(H)$ contains a subgroup isomorphic to $N_G(H)/C_G(H)$.
2. a. Prove that any finite p -group has a nontrivial center.

b. Let G be a finite group. Suppose that for each prime $p \mid |G|$, the Sylow p -subgroup of G is normal. Prove that G has a nontrivial center. Hint: First show G is an internal direct product of its Sylow subgroups.
3. Let $G = GL_2(q)$ where $q = p^a$ for some prime p .
 - a. Determine the order of G .
 - b. Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in G$. Determine the size of the conjugacy class of A .
4. Prove there is no simple group of order 56.
5. Let $H \trianglelefteq G, K \trianglelefteq G$, and $G = HK$. Prove that:

$$G/(H \cap K) \cong G/K \times G/H.$$

6. Let $H \leq G, K \leq G$. Prove $HK \leq G$ if and only if $HK = KH$.
7. Suppose $f : G \rightarrow A$ is a group homomorphism, where A is abelian. Prove that any subgroup of G which contains $\ker f$ is normal.