1. $H \trianglelefteq N_G(H)$ so $N_G(H)$ acts on H by conjugation. This gives a homomorphism

$$\varphi : N_G(H) \longrightarrow \operatorname{Aut} H \qquad \varphi(g) = i_g$$

$$\text{Kernel of } \varphi = \{ g \in N_G(H) \mid ghg^{-1} = h \ \forall h \in H \}$$
$$= \{ g \in N_G(H) \mid gh = hg \ \forall h \in H \} = C_G(H).$$

Thus $N_G(H)/C_G(H) \cong \text{image } \varphi \le \operatorname{Aut} H.$

2a. The class equation gives $|P| = |Z(P)| + \sum [P : C_P(x_i)]$

with the sum over conj. classes of reps of noncentral elements. Since $|P| = p^a$ and $C_P(x_i) \neq P$ then $p \mid [P : C_P(x_i)]$ and $p \mid |P|$ so $p \mid |Z(P)|$ so $|Z(P)| > 1$.

b. Let $P_1, \ldots, P_t$ be Sylows for the distinct primes dividing $|G|$.

Then $|G| = |P_1| \cdot |P_2| \cdots |P_t|$. Lagrange's Theorem implies $|P_i \cap P_j| = e$ for $i \neq j$. Thus $i \neq j \Rightarrow P_i P_j \cong P_i \times P_j$ by HW exercise. So

$$G \cong P_1 \times \cdots \times P_t. \quad \text{But then}$$

$$Z(G) = Z(P_1) \times \cdots \times Z(P_t) \quad \text{since the products}$$
in $P_1 \times \cdots \times P_t$ commute with each other.

Thus $Z(G) \neq \{e\}$ by 2a.

3a. $|G| = (q^2-1)(q^2-q)$

b. $|\text{class } A| = [G : C_G(A)]$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} A = \begin{pmatrix} a & a+b \\ c & c+d \end{pmatrix}$$

$$(A)\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+c & b+d \\ c & d \end{pmatrix}$$

Thus $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in C_G(A) \Rightarrow c=0,\ a=d$

So $$C_G(A) = \left\{ \begin{pmatrix} a & b \\ c & a \end{pmatrix} \middle| a \neq 0 \right\}$$

$$|C_G(A)| = (q-1)q = q^2-q$$

$$|\text{class } A| = \frac{(q^2-1)(q^2-q)}{q^2-q} = \boxed{q^2-1}$$

4. $56 = 2^3 \cdot 7$ Let $n_7 = \#$ Sylow 7 subgroups. Then

$n_7 \equiv 1 \bmod 7$, $n_7 \mid 8$ so $n_7 = 1$ or 7

Suppose $n_7 = 8$. This means, since distinct Sylows must have trivial $\cap$ as they have prime order, That there are $8 \cdot 6 + 1 = 49$ elements in the Sylow 7's the identity and 48 elements of order 7. This leaves only 7 elements left. Together with $\{e\}$ they must form the unique Sylow 2-subgroup. Thus $n_7 \neq 1 \Rightarrow n_2 = 1$, So either the Sylow 7 or Sylow 2 subgroup is normal. Thus G cannot be simple.

5. Define $\varphi : G \longrightarrow G/K \times G/H$ by

$$\varphi(g) = (gK, gH)$$

Notice that $\ker \varphi = H \cap K$. Thus

$$G/H\cap K \cong \text{image } \varphi.$$

Let's consider the lattice:

$$\begin{tikzcd}[column sep=small]
 & G = HK \arrow[dl, dash] \arrow[dr, dash] & \\
H \arrow[dr, dash] & & K \arrow[dl, dash] \\
 & H\cap K \arrow[d, dash] & \\
 & e &
\end{tikzcd}$$

$$|G/K| = |HK/K| = |H/H\cap K|$$
$$|G/H| = |HK/H| = |K/H\cap K|$$

$$\begin{aligned}
|G/H\cap K| &= [G:H]\,[H:H\cap K] \\
&= |HK/K| \cdot |H|/|H\cap K| \\
&= \frac{|K|}{|K\cap H|} \cdot \frac{|H|}{|H\cap K|}
\end{aligned}$$

Thus $|G/H\cap K| = |G/K|\,|G/H| = |G/K \times G/H|$

so image φ is everything. //

6. Suppose $HK \leq G$. Let $hk \in HK$. Then

$(hk)^{-1} = k^{-1}h^{-1} \in HK$ so

$k^{-1}h^{-1} = \tilde{h}\tilde{k}$. Thus everything of

the form kh is also of the form $\hat{h}\hat{k}$ and vice versa

so $HK = KH$.

Now suppose $HK = KH$.

Let $h_1k_1,\ h_2k_2 \in HK$

$$(h_1k_1)(h_2k_2)^{-1} = h_1k_1k_2^{-1}h_2^{-1}$$

$$= h_1\,(k_1k_2^{-1}h_2^{-1})$$

$= h_1 \cdot \hat{h}\hat{k}$ since $k_1k_2^{-1}h_2^{-1} \in KH = HK$

$\in HK$.

Thus $HK \leq G$.

7. $G/\ker f \cong$ Image f is abelian.

Thus any subgroup of $G/\ker f$ is normal.

Thus any subgroup of G containing

$\ker f$ is normal by the correspondence Theorem.