

1. $H \trianglelefteq N_G(H)$ so $N_G(H)$ acts on H by conjugation. This gives a homomorphism

$$\psi: N_G(H) \rightarrow \text{Aut } H \quad \psi(g) = i_g$$

$$\begin{aligned} \text{Kernel of } \psi &= \{g \in N_G(H) \mid ghg^{-1} = h \ \forall h \in H\} \\ &= \{g \in N_G(H) \mid gh = hg \ \forall h \in H\} = C_G(H) \end{aligned}$$

$$\text{Thus } N_G(H)/C_G(H) \cong \text{image } \psi \leq \text{Aut } H.$$

2a. The class equation gives $|P| = |Z(P)| + \sum [P : C_P(x_i)]$

with the sum over ^{conj.} classes of reps of noncentral elements. Since $|P| = p^a$ and $C_P(x_i) \neq P$ then $p \mid [P : C_P(x_i)]$ and $p \mid |P|$ so $p \mid |Z(P)|$ so $|Z(P)| > 1$.

b. Let P_1, \dots, P_e be Sylows for the distinct primes dividing $|G|$.

Then $|G| = |P_1| \cdot |P_2| \cdot \dots \cdot |P_e|$. Lagrange's Theorem implies $|P_i \cap P_j| = e$ for $i \neq j$. Thus $i \neq j \Rightarrow P_i P_j \cong P_i \times P_j$ by HW exercise. So

$$G \cong P_1 \times \dots \times P_e. \quad \text{But then}$$

$Z(G) = Z(P_1) \times \dots \times Z(P_e)$ since the products in $P_i \times \dots \times P_e$ commute with each other.

Thus $Z(G) \neq \{e\}$ by 2a.

$$3a. |G| = (q^2 - 1)(q^2 - q)$$

$$b. |\text{class } A| = [G : C_G(A)]$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} A = \begin{pmatrix} a & arh \\ c & crd \end{pmatrix}$$

$$A \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} arc & brd \\ c & d \end{pmatrix}$$

$$\text{Thus } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in C_G(A) \Rightarrow c=0, a=d$$

$$\text{So } C_G(A) = \left\{ \begin{pmatrix} a & b \\ c & a \end{pmatrix} \mid a \neq 0 \right\}$$

$$|C_G(A)| = (q-1)q = q^2 - q$$

$$|\text{class } A| = \frac{(q^2 - 1)(q^2 - q)}{q^2 - q} = \boxed{q^2 - 1}$$

4. $56 = 2^3 \cdot 7$. Let $n_7 = \#$ Sylow 7 subgroups. Then

$$n_7 \equiv 1 \pmod{7}, n_7 \mid 8 \text{ so } n_7 = 1 \text{ or } 7$$

Suppose $n_7 = 8$. This means, since distinct Sylows must have trivial \cap as they have prime order,

that there are $8 \cdot 6 + 1 = 49$ elements in the Sylow 7's

the identity and 48 elements of order 7. This leaves only 7 elements left. Together with \mathbb{Z}_3 they must form the unique Sylow 2-subgroup. Thus $n_7 \neq 1 \Rightarrow n_2 = 1$, so either the Sylow 7 or Sylow 2 subgroup is normal. Thus G cannot be simple.

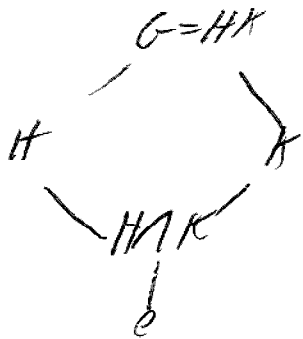
5. Define $\psi: G \rightarrow G/K \times G/H$ by

$$\psi(g) = (gK, gH).$$

Notice that $\ker \psi = HK$. Thus

$$G/HK \cong \text{image } \psi.$$

Let's consider the lattice:



$$|G/K| = |HK/K| = |H/HK|$$

$$|G/H| = |HK/H| = |K/HK|$$

$$|G/HK| = [G:H] [H/HK]$$

$$= |HK/K| \cdot |H/HK|$$

$$= \frac{|K|}{|HK|} \cdot \frac{|H|}{|HK|}$$

$$\text{Thus } |G/HK| = |G/K| |G/H| = |G/K \times G/H|$$

so image ψ is everything. //

6. Suppose $HK \leq G$. Let $hk \in HK$. Then

$$(hk)^{-1} = k^{-1}h^{-1} \in HK \text{ so}$$

$$k^{-1}h^{-1} = \tilde{h}\tilde{k}. \text{ This is of the form } \tilde{h}\tilde{k} \text{ and vice versa}$$

so $HK = KH$.

Now suppose $HK = KH$.

Let $h_1k_1, h_2k_2 \in HK$

$$(h_1k_1)(h_2k_2)^{-1} = h_1k_1k_2^{-1}h_2^{-1}$$

$$= h_1(k_2^{-1}h_2^{-1})$$

$$= h_1\tilde{h}\tilde{k} \text{ since } k_2^{-1}h_2^{-1} \in KH = HK \in HK.$$

Thus $HK \leq G$.

7. $G/\text{Ker } f \cong \text{Image } f$ is abelian.

Thus any subgroup of $G/\text{Ker } f$ is normal.

Thus any subgroup of G containing

$\text{Ker } f$ is normal by the Correspondence Theorem.