

**Math 8300 Fall 2006 Midterm Exam #2.**

**Instructions:** You must do problem 1. Then choose 4 of the remaining 7 problems for a total of five problems. Clearly label in your blue book which problems you wish graded.

1. Completely classify groups of order 20, including a presentation for each isomorphism type and a proof that your list is complete.
- 2 a. State three equivalent conditions for a finite group  $G$  to be nilpotent.  
b. Is  $S_4$  solvable? Explain.
3. Give a finite presentation for  $A_4$  using two generators.
4. Suppose  $H$  and  $K$  are normal, nilpotent subgroups of a group  $G$  of nilpotence class  $h$  and  $k$  respectively, and suppose  $G = HK$ . Prove that  $G$  is nilpotent of class at most  $h + k$ .
5. Explain the construction of a free group  $F(S)$  on a set  $S$ . Then state precisely the universal property which characterizes this group.
6. Let  $K, H, N$  be nontrivial normal subgroups of  $G$  such that  $G = H \times K$ . Prove that  $N$  is in the center of  $G$  or that  $N$  intersects one of  $H$  and  $K$  nontrivially.
7. Suppose that  $H \trianglelefteq G$  and  $H \cap G' = \{e\}$ . Prove that  $H \leq Z(G)$ .
8. You are given  $G, H, K$ . Either prove that  $G \cong H \times K$  for some  $K$  or demonstrate that it is not.
  - a.  $G = GL_n(F), H = SL_n(F)$  where  $F$  is a field.
  - b.  $G = S_n, H = A_n$ .
  - c.  $G = Q_8, H = Z_4$ .